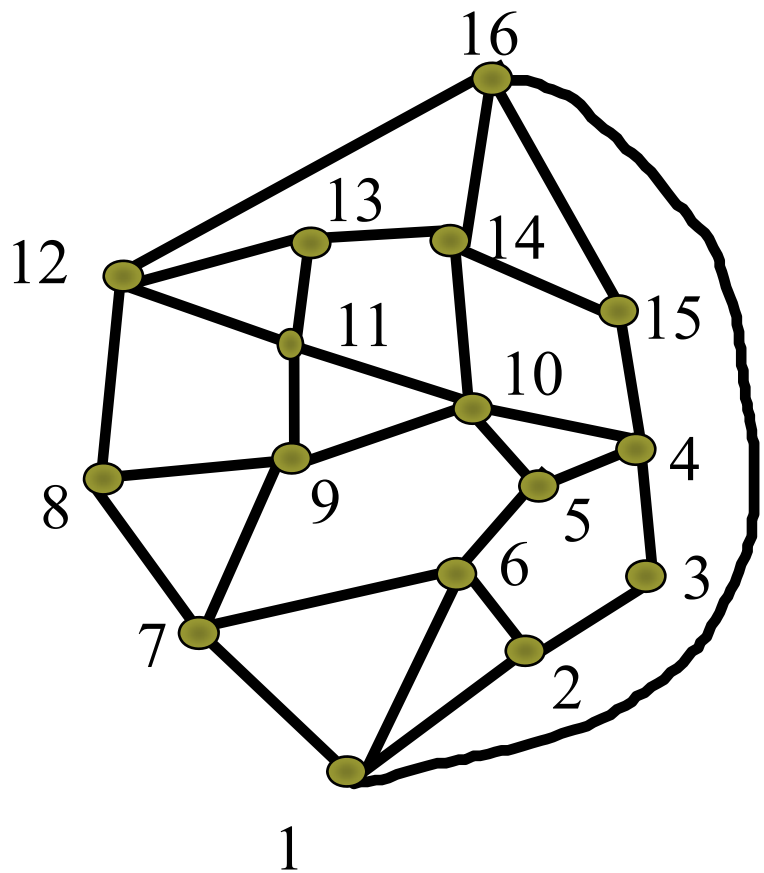
Various Orders and Drawings of Plane Graphs

Takao Nishizeki

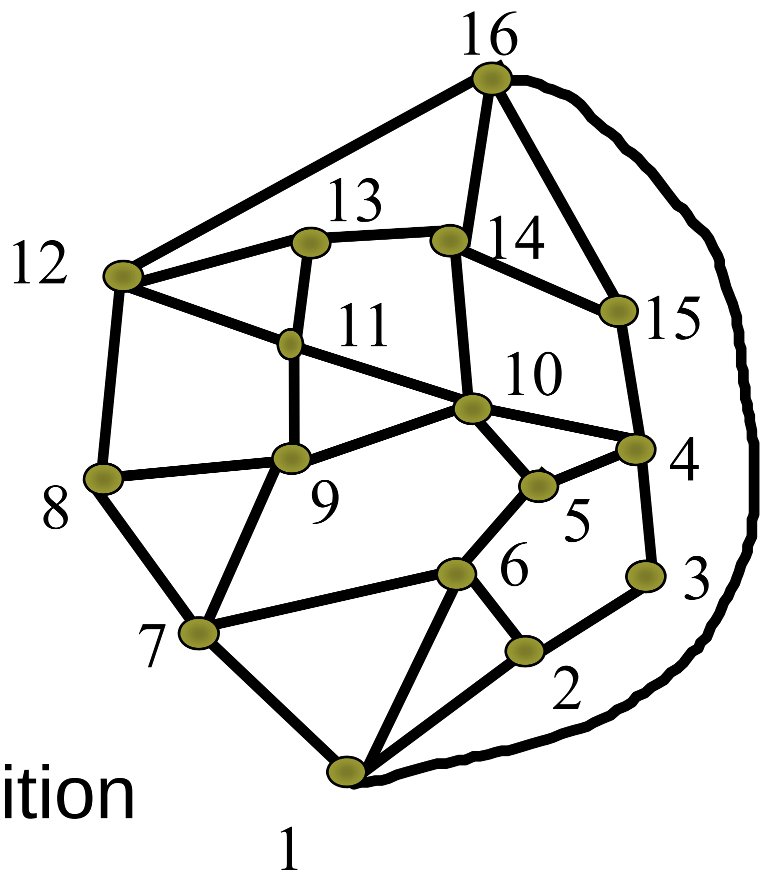
Tohoku University

Vertex ordering



Vertex ordering

- *st*-numbering
- Canonical ordering
- 4-canonical ordering
- Canonical decomposition
- 4-canonical decomposition



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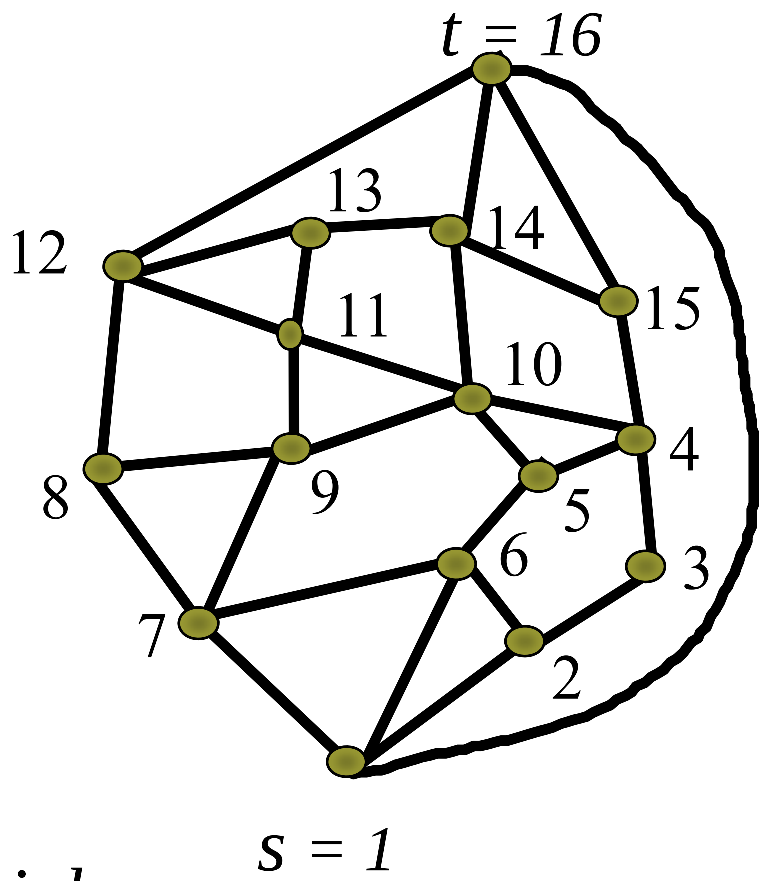
st-numbering

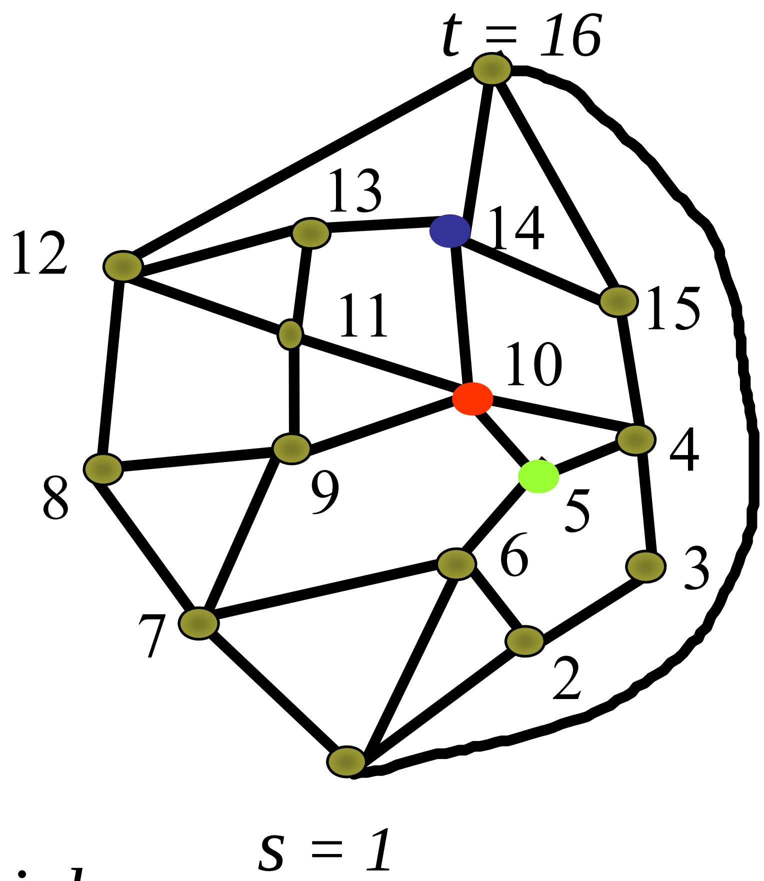


$i \neq s,t$ has two neighbors j, k

$$j < i < k.$$

st-numbering

$$i = 10$$

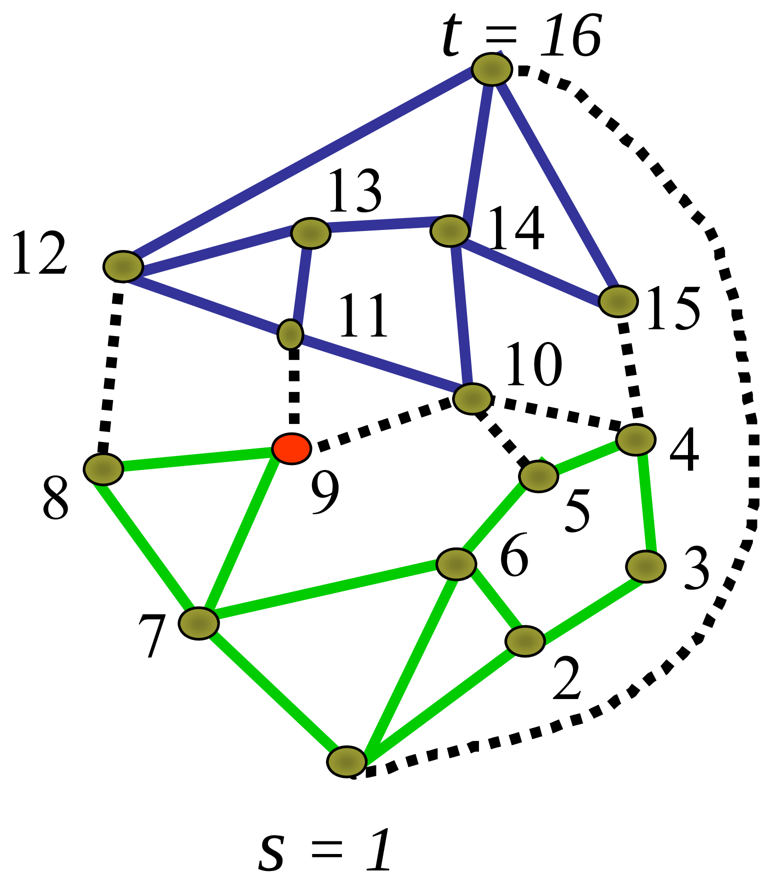


$i \neq s, t$ has two neighbors j, k

$$j < i < k.$$

st-numbering

$$i = 9$$



For any i, both vertices $\{1, 2, \cdots, i\}$ and $\{i+1, i+2, \cdots, n\}$ induce connected subgraphs.

Application of *st*-numbersing

Planarity testing

Visibility drawing

Internet routing

Vertex ordering

- *st*-numbering
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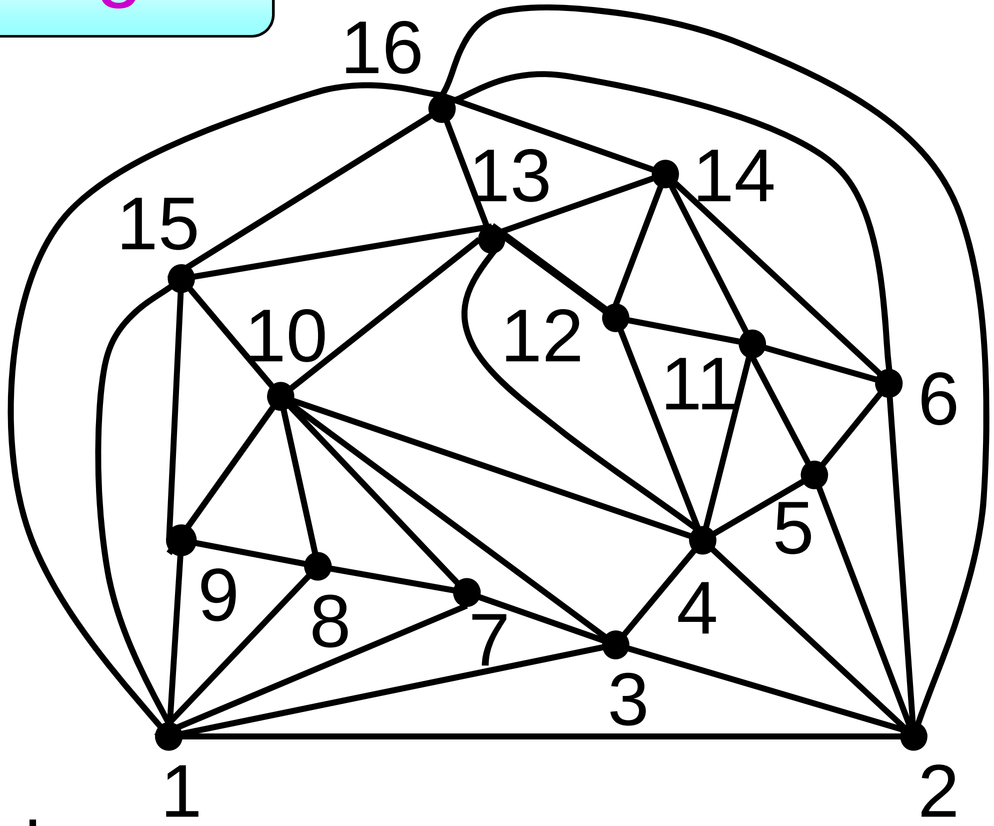
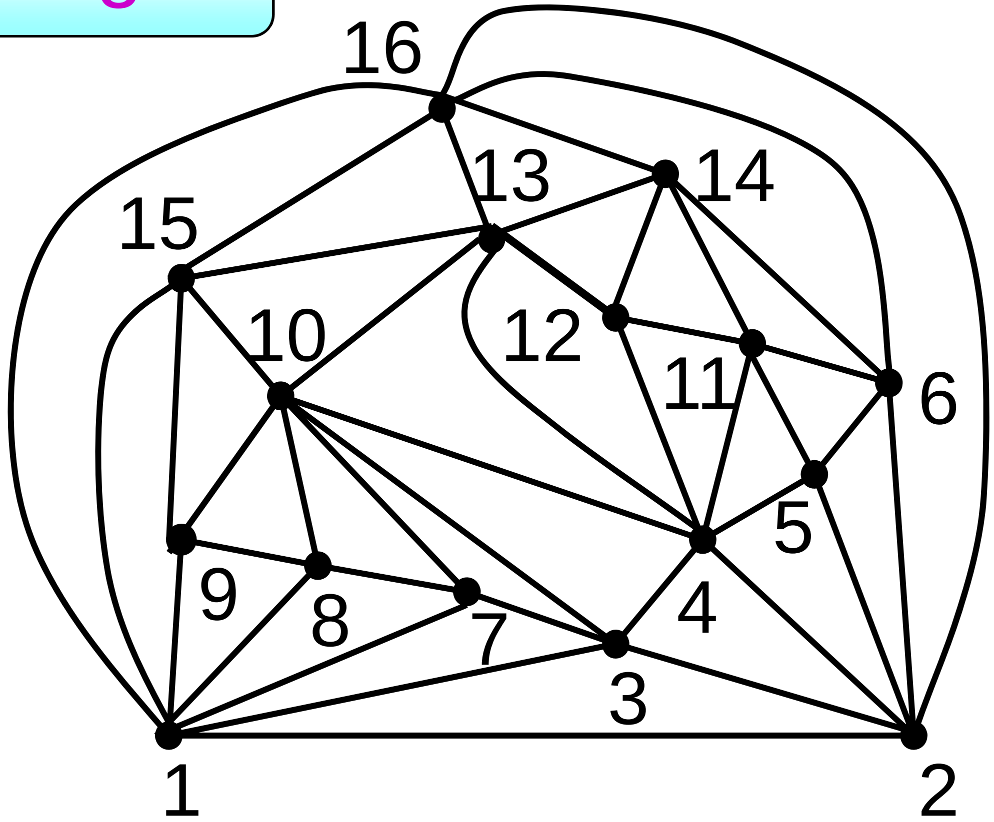
- 4-canonical ordering
- Canonical decomposition
- 4-canonical decomposition

Canonical Ordering



Triangulated plane graph

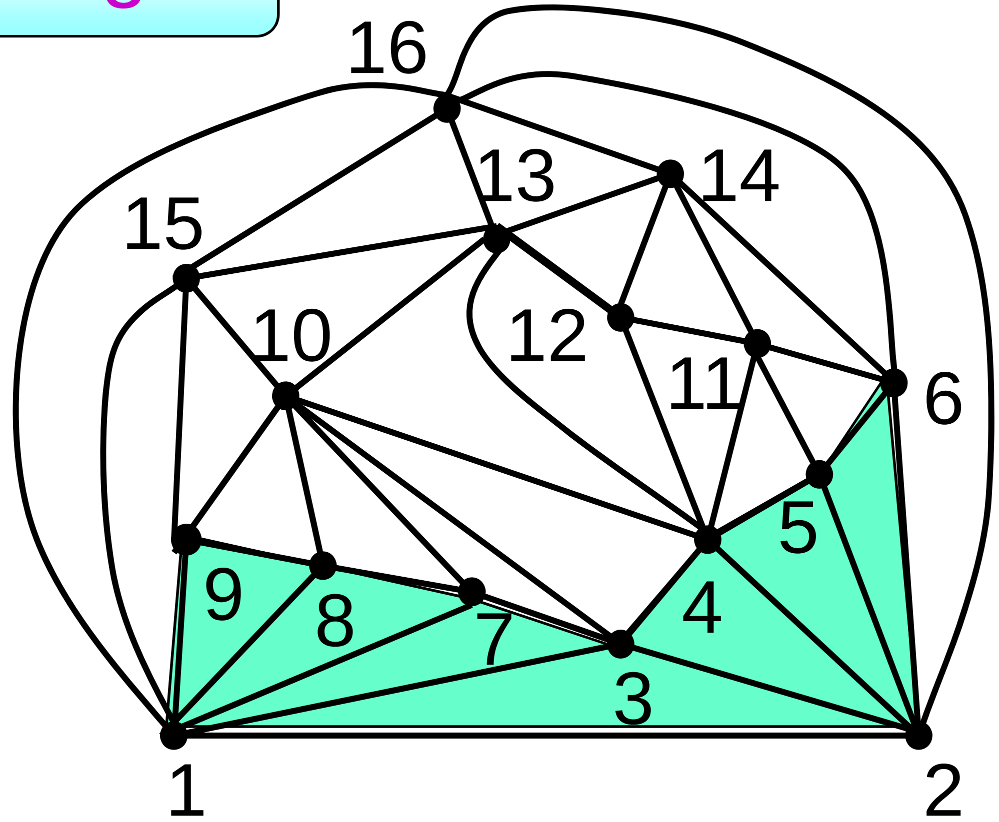
Canonical Ordering



G_k : subgraph of G induced by vertices $1, 2, \cdots, k$

Canonical Ordering

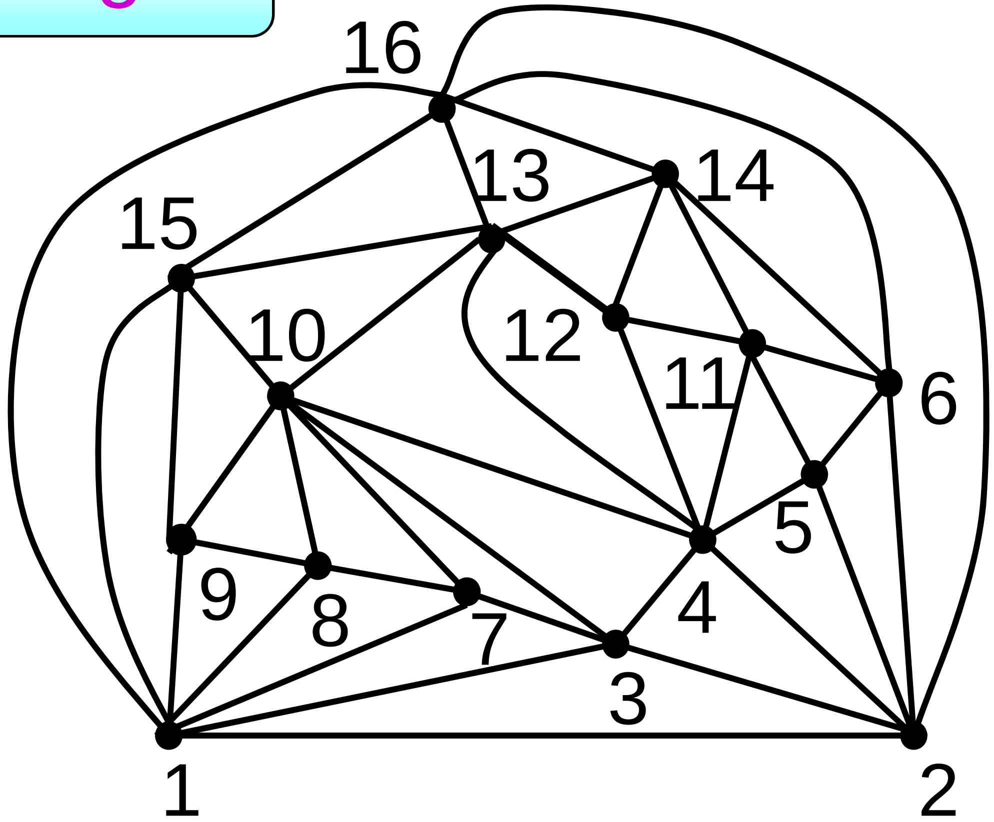
G_9



G_k : subgraph of G induced by vertices $1, 2, \cdots, k$

Canonical Ordering

For any $k, 3 \leq k \leq n$



(co1) G_k is biconnected and internally triangulated

(co2) vertices 1 and 2 are on the outer face of G_k

(co3) vertex k+1 is on the outer face of G_k and the neighbor of k+1 is consecutive on the outer cycle $C_o(G_k)$.

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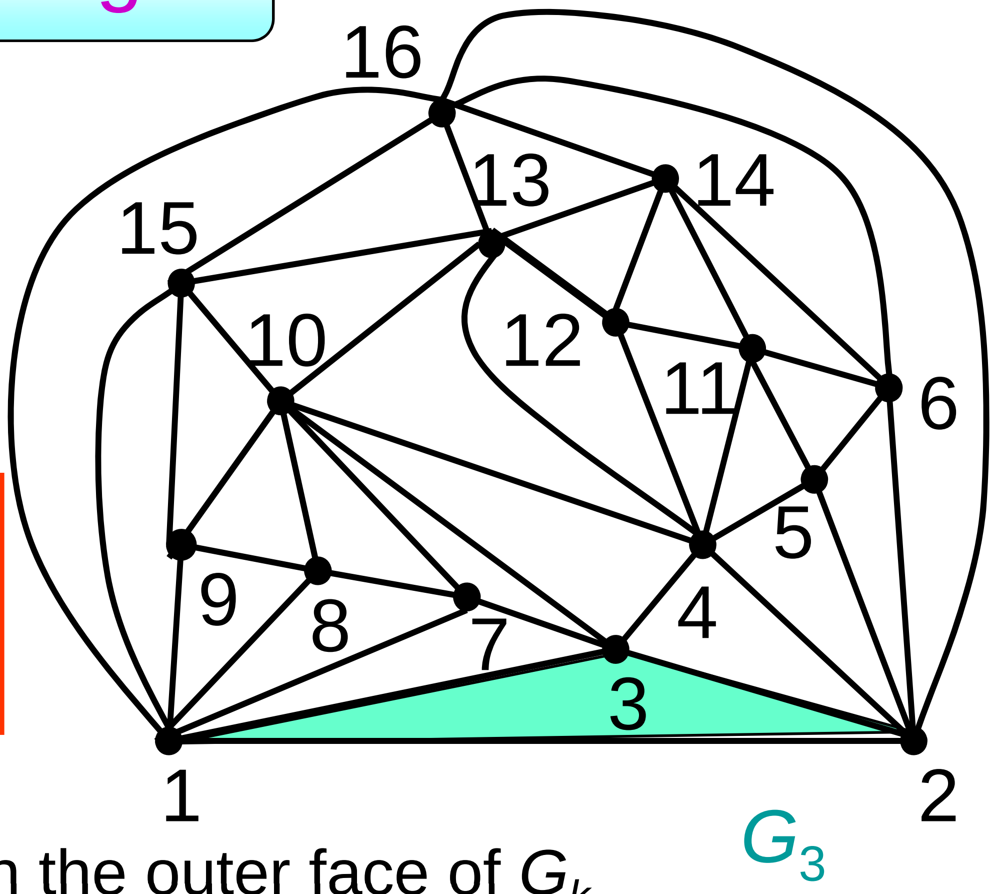

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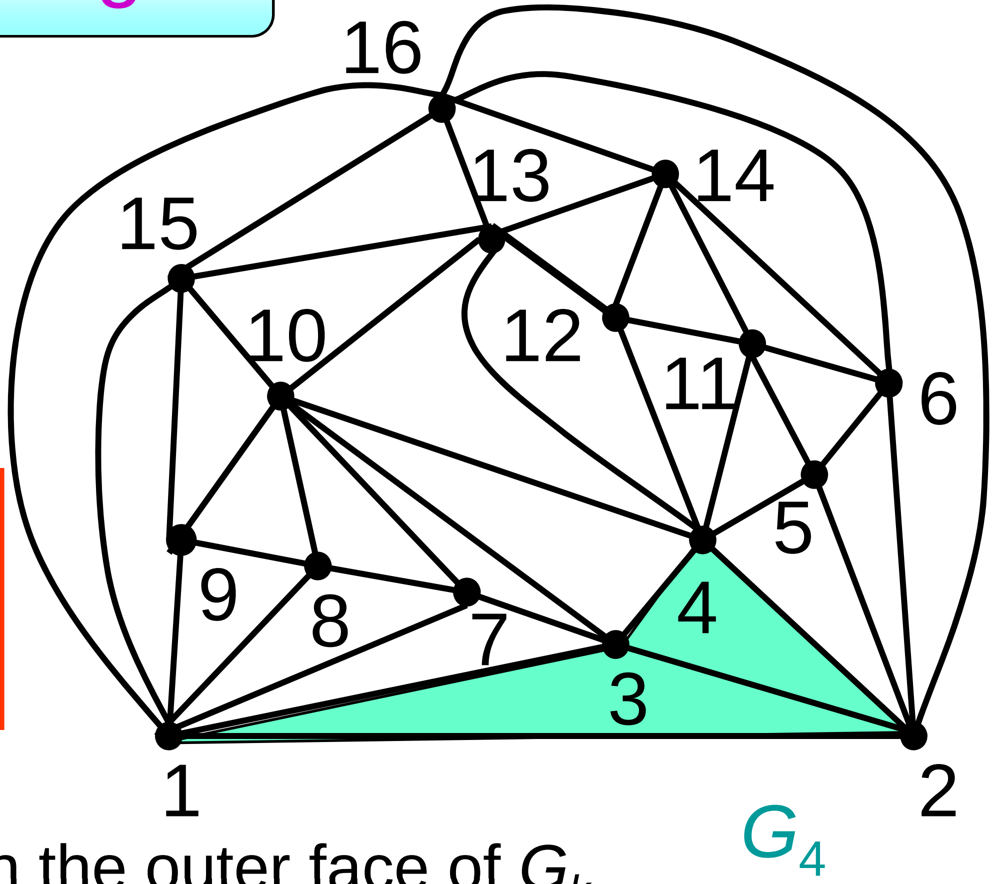

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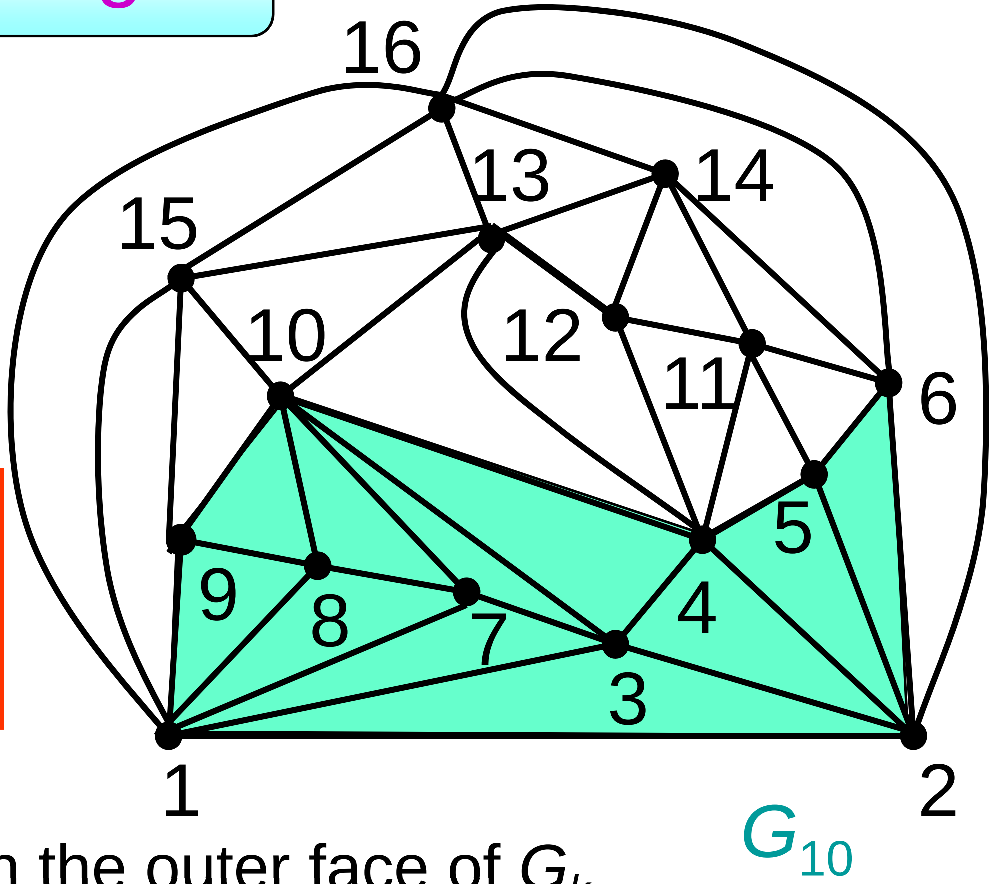

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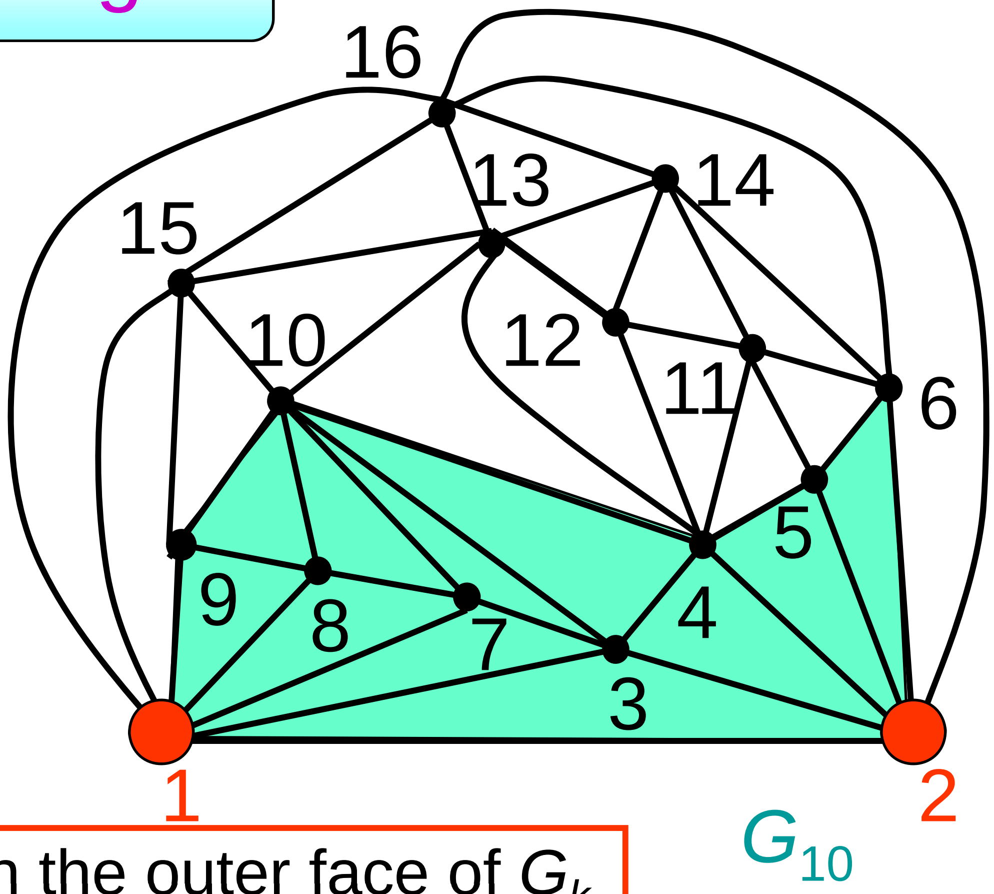

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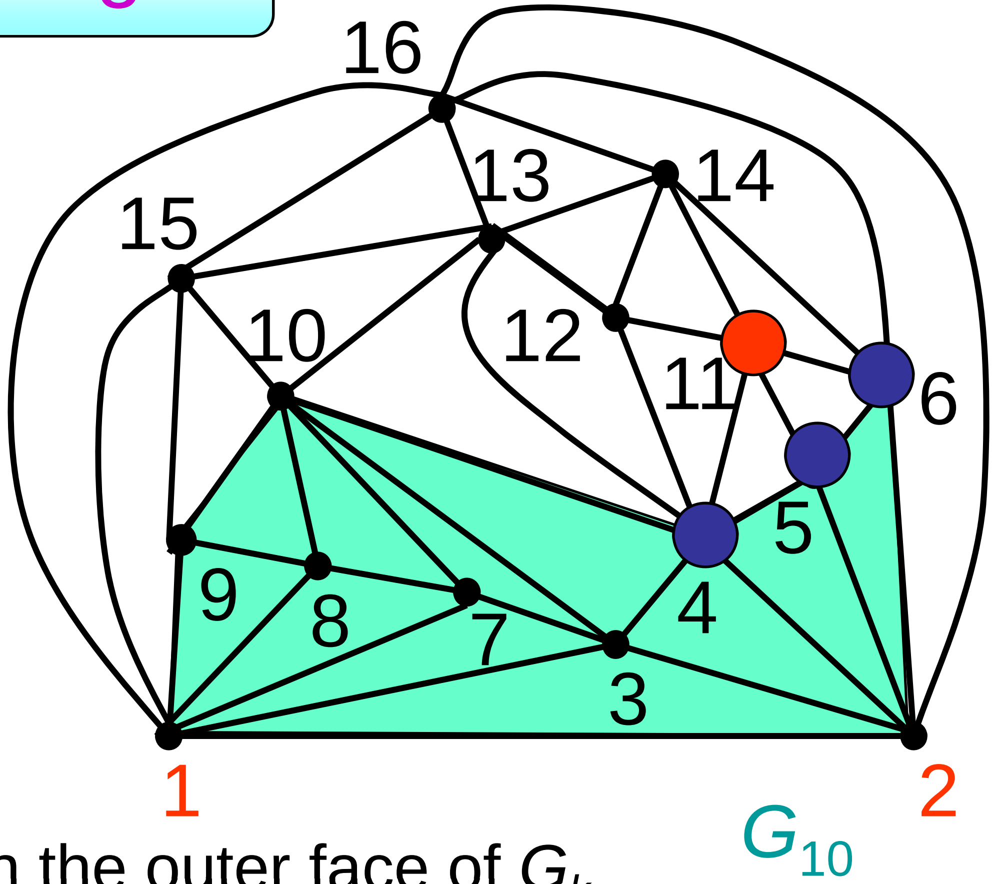

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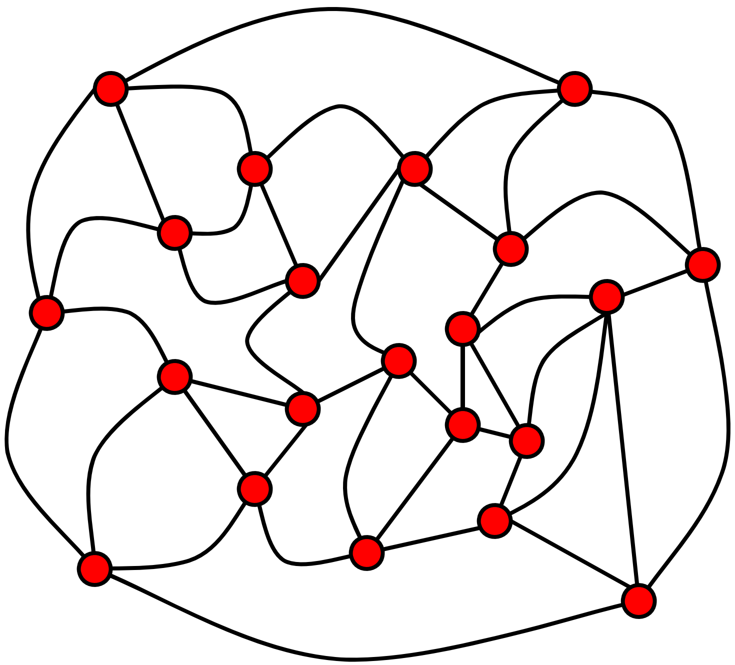
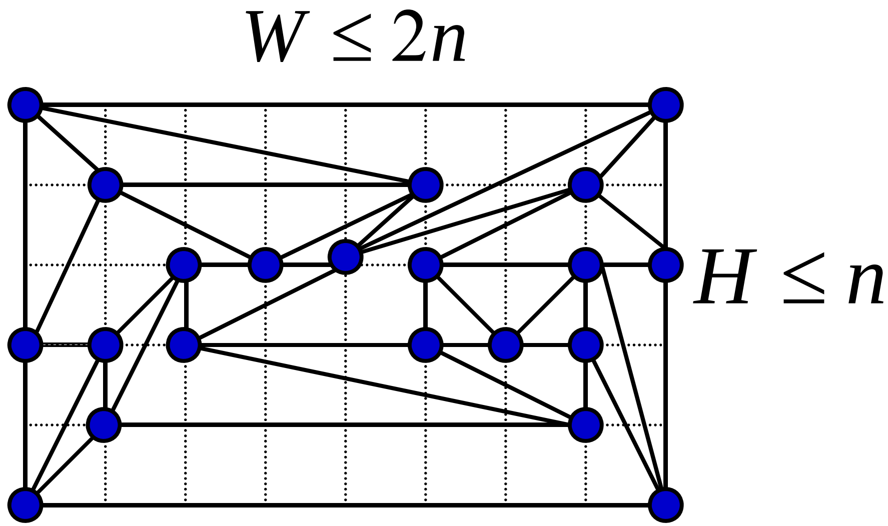
Straight Line Grid Drawing



Plane graph

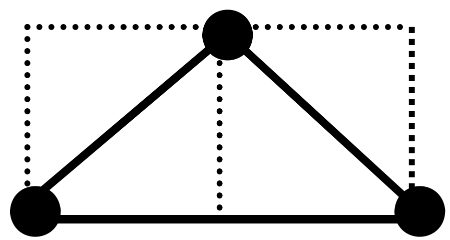


Straight line grid drawing.

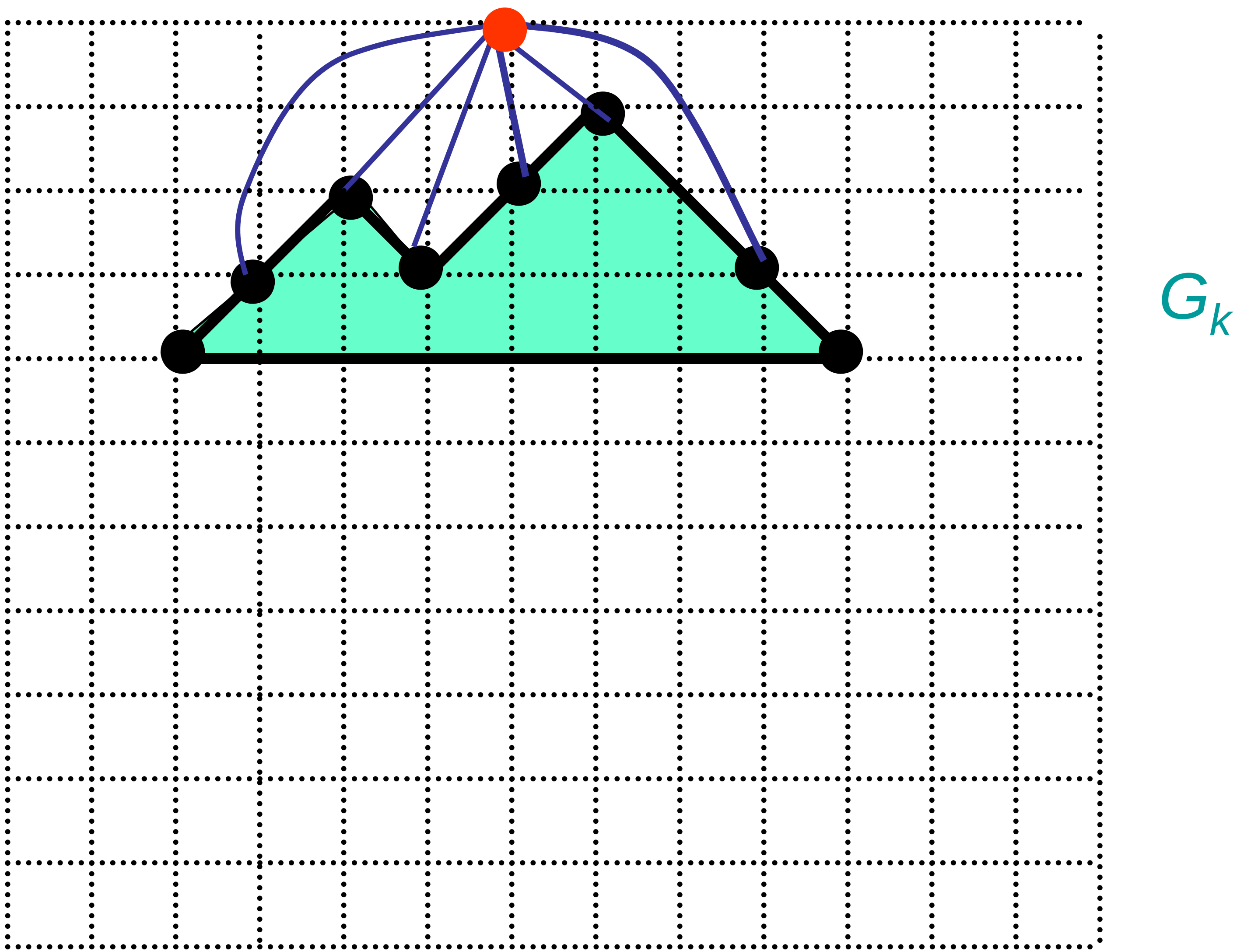
de Fraysseix *et al.* '90

$$W \times H \leq 2n^2$$

Initial drawing of G_3

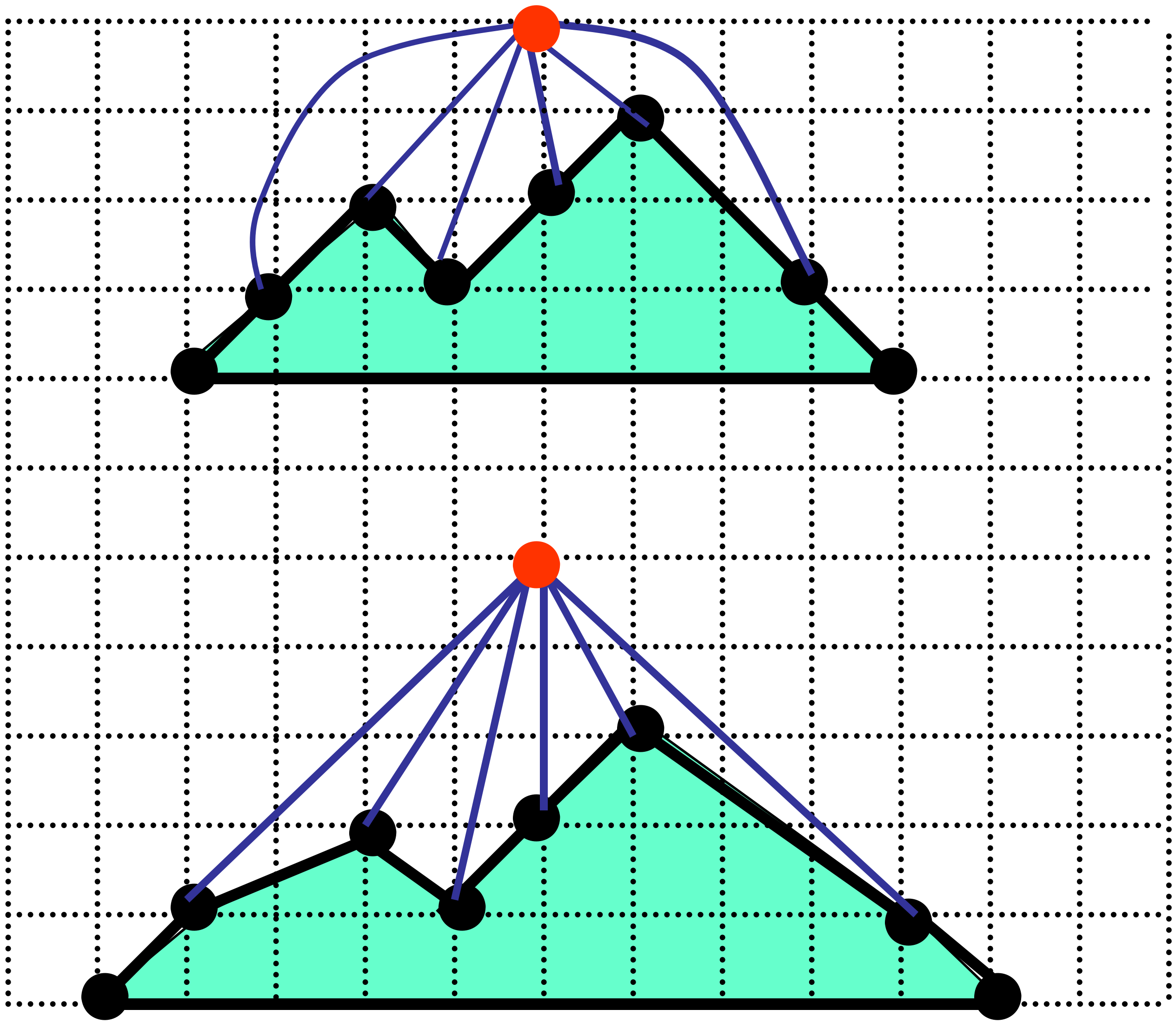


Install $k+1$

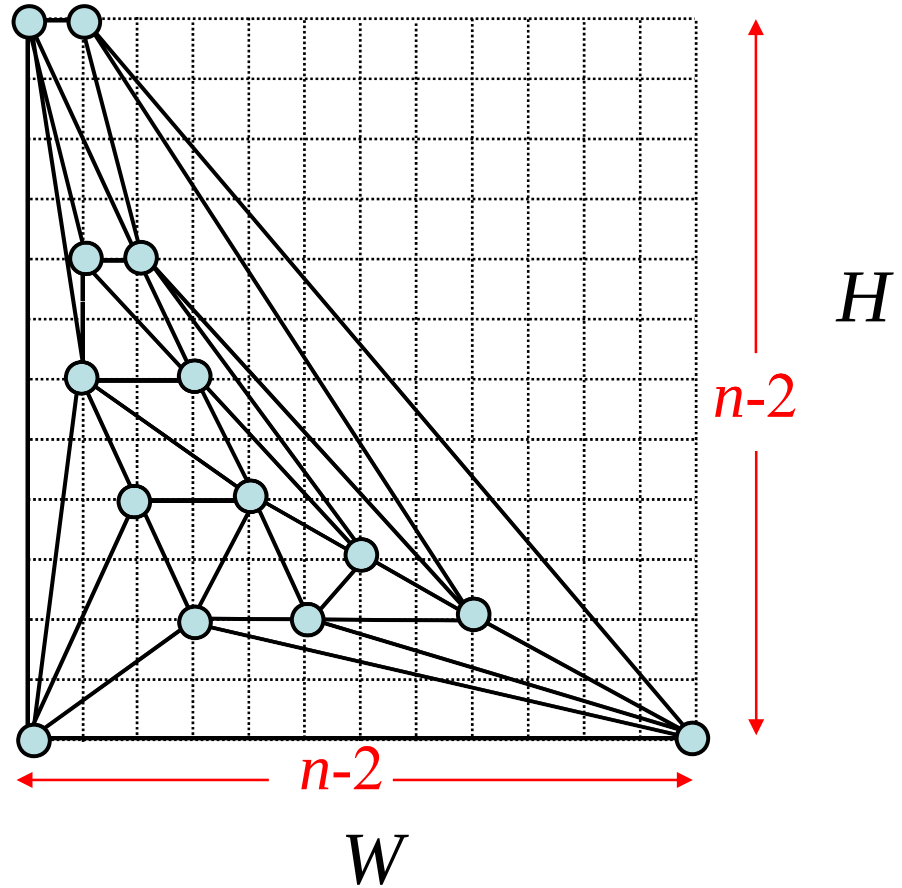


Shift method

Shift and install $k + 1$



Schnyder '90

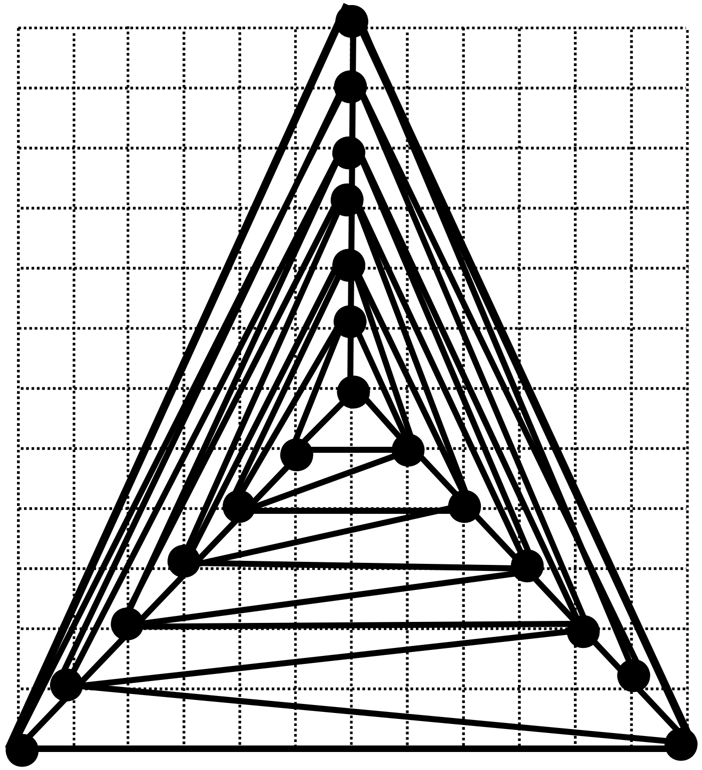


$$W \times H \leq n^2$$

Upper bound

What is the minimum size of a grid required for a straight line drawing?

Lower Bound



$$H \geq \frac{2}{3}n$$

$$W \geq \frac{2}{3}n$$

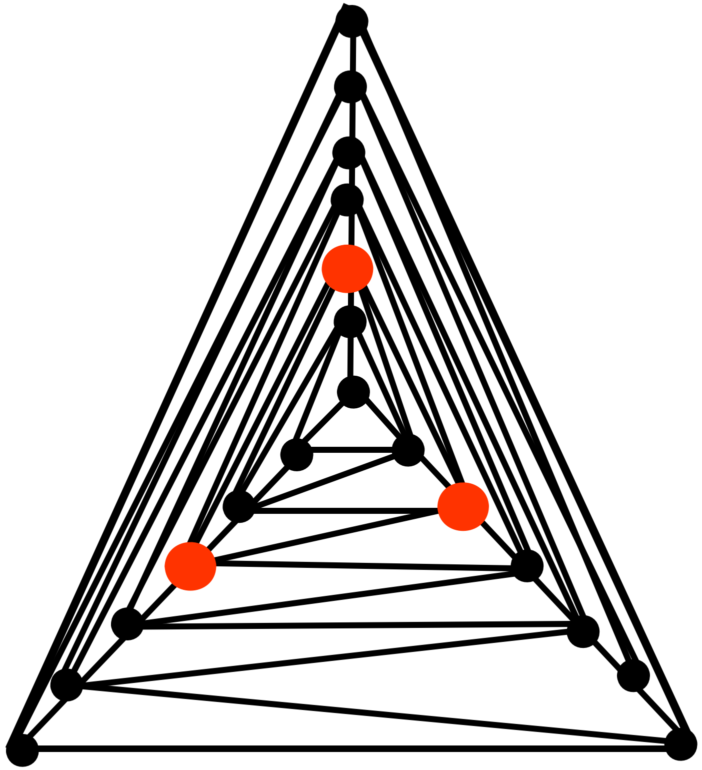
$$W \times H \geq \left(\frac{2}{3}n\right)^2$$

A restricted class of plane graphs may have more compact grid drawing.

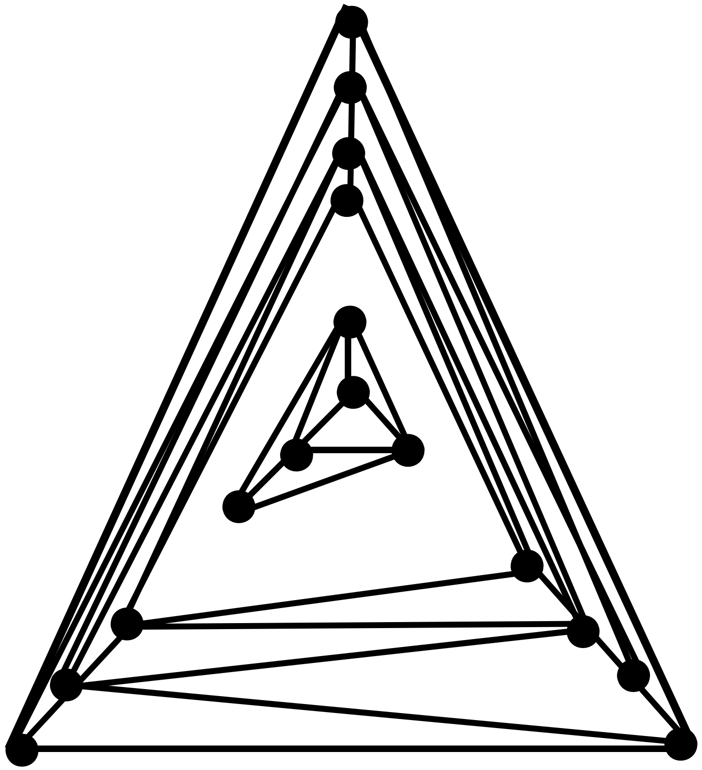
Triangulated plane graph

3-connected graph

4-connected ?



not 4-connected



disconnected

How much area is required for 4-connected plane graphs?

Straight line grid drawing

Miura *et al.* '01

Input: 4-connected plane graph G

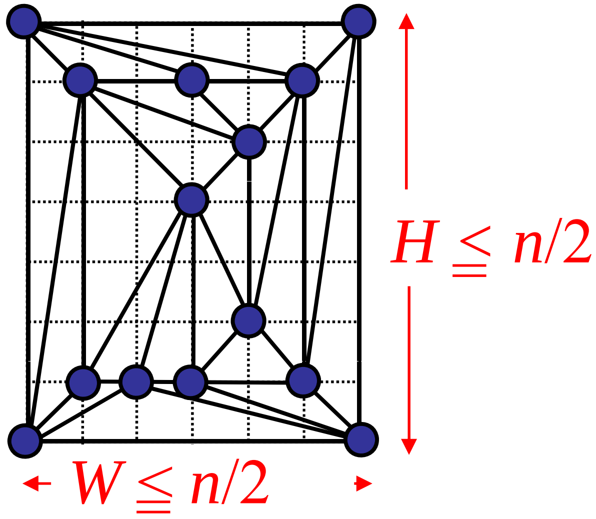
Output: a straight line grid drawing

Grid Size :

$$W, H \leq \frac{n}{2}$$

Area:

$$W \times H \leq \frac{n^2}{4}$$

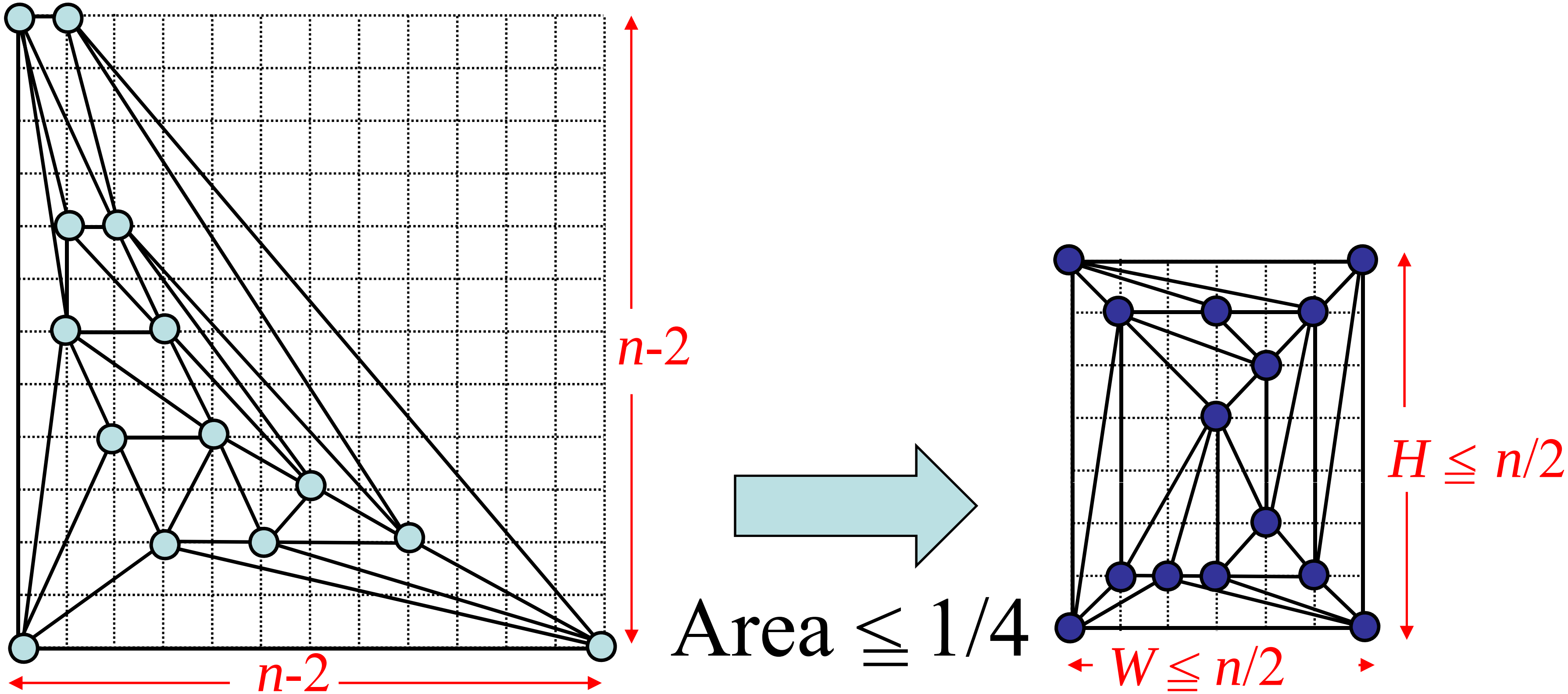


Schnyder '90

plane graph G

Miura *et al.* '01

4-connected plane graph G



Area$\fallingdotseq n^2$

Area$\leqq n^2/4$

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Main idea

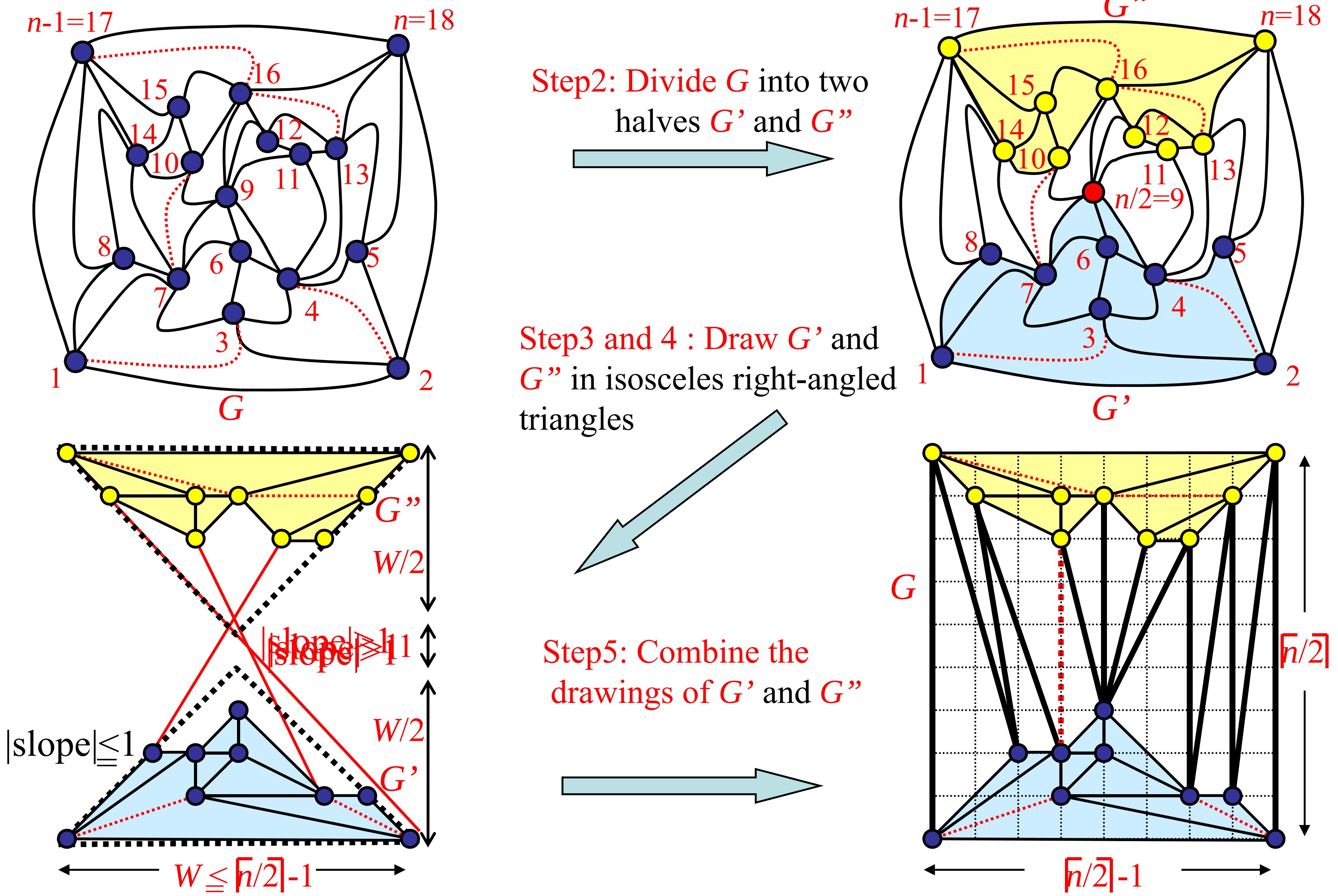
Triangulate all inner faces

Step1: find a 4-canonical ordering

Step2: Divide G into two halves G' and G''

Step3 and 4 : Draw G' and G'' in isosceles right-angled triangles

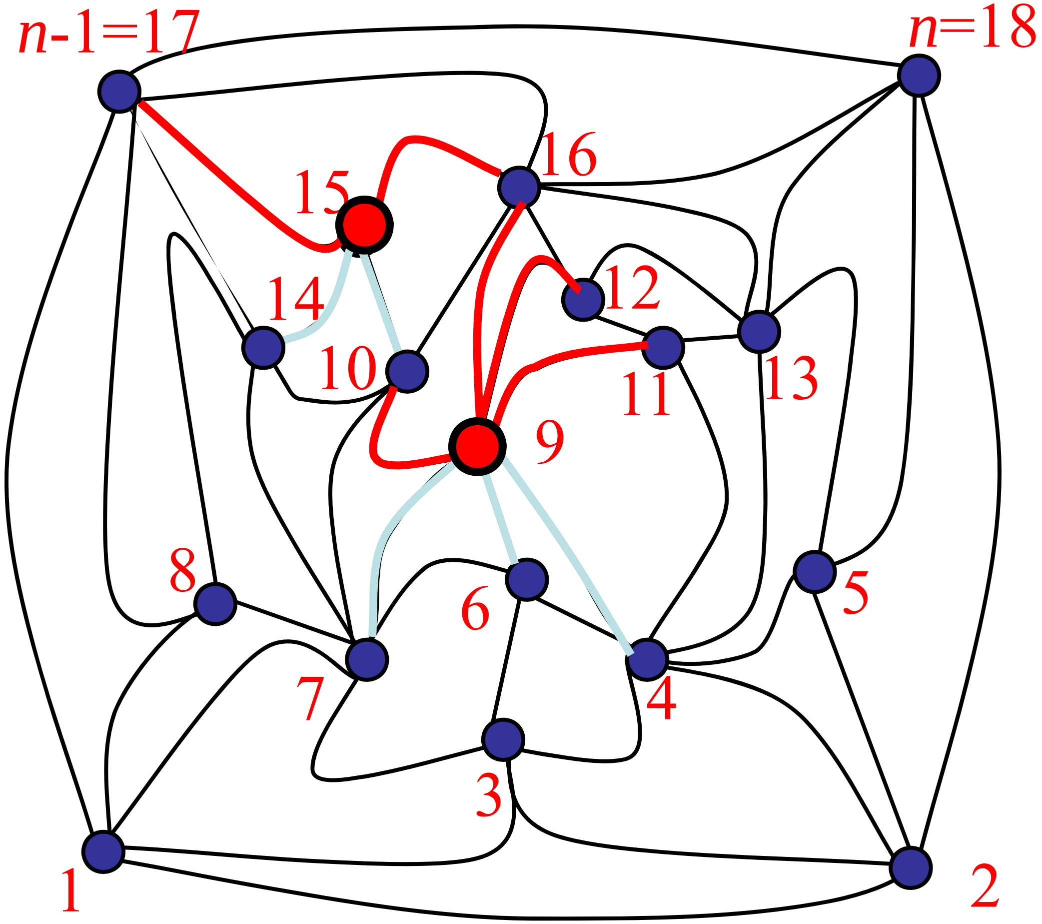
Step5: Combine the drawings of G' and G''



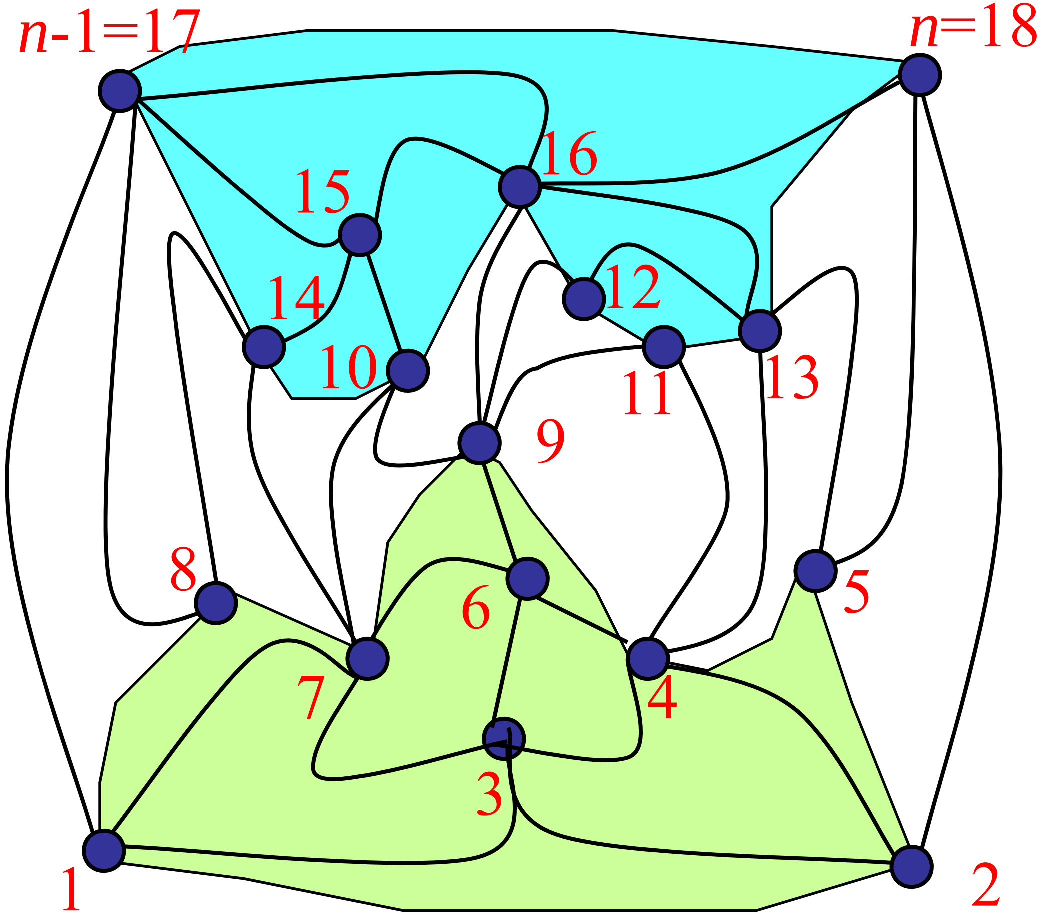
4-canonical ordering [KH97](4-connected graph)

(1) Edges (1,2) and (n,n-1) are on the outer face

(2) For each vertex k, $3 < k < n-2$,
at least two neighbors have lower number and at least two neighbors have higher neighbor.



4-canonical ordering [KH97](4-connected graph)

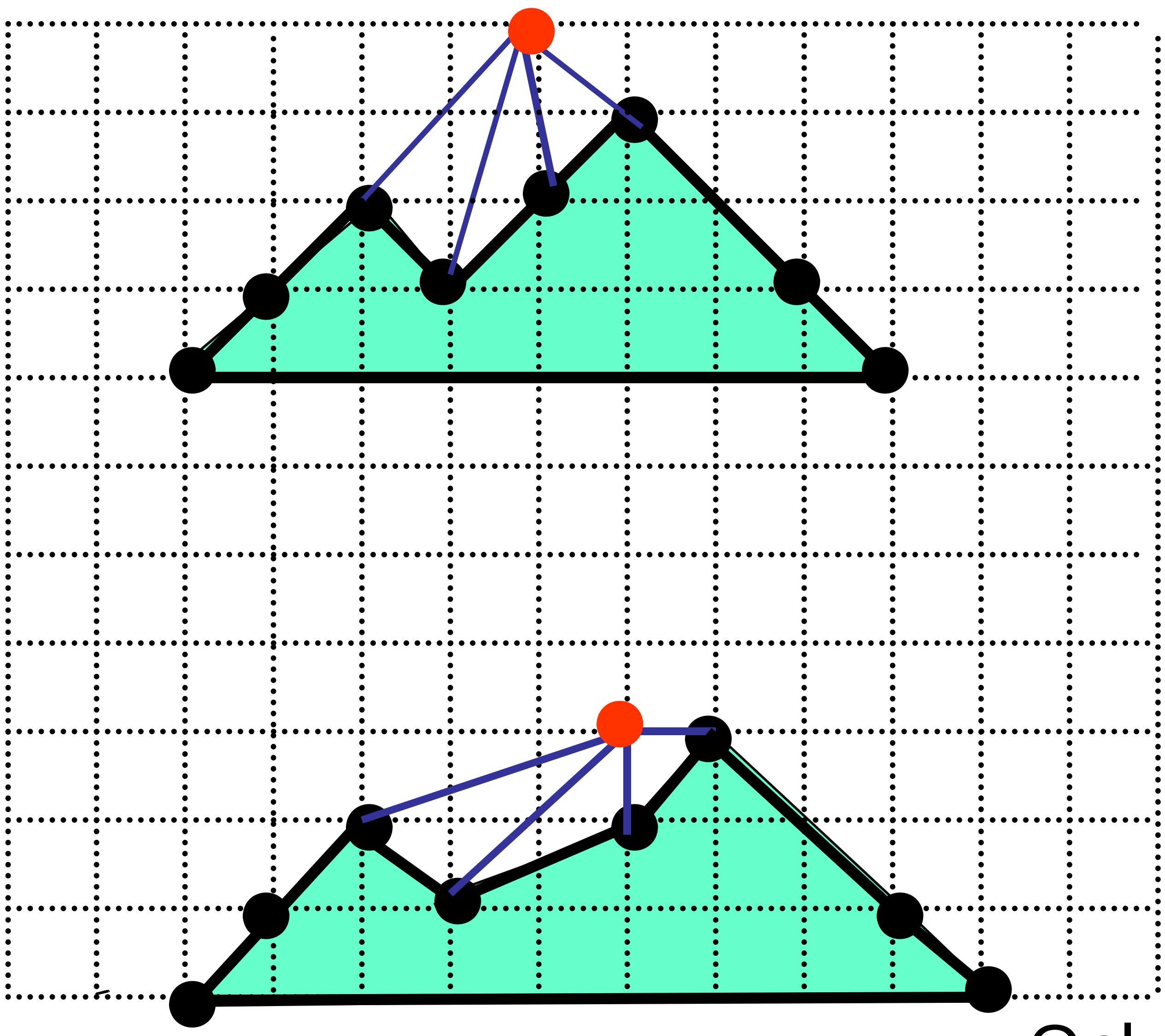


Generalization of an *st*-numbering

Both vertices $\{1, 2, \cdots, i\}$ and $\{i+1, i+2, \cdots, n\}$ induce 2-connected subgraphs.

Shift method

Shift and install $k + 1$



Only one shift

Shift method

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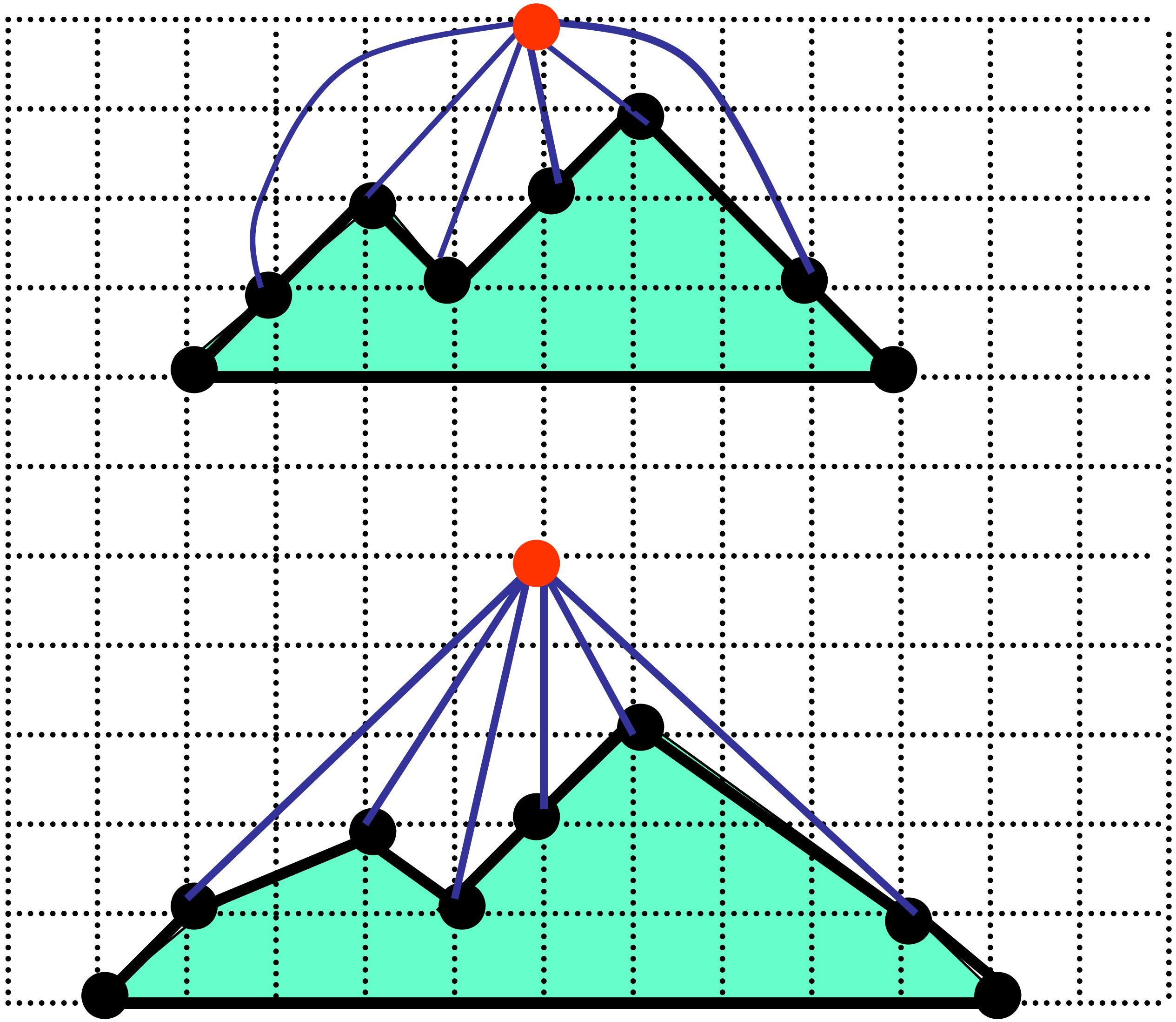

Graph is not triangulated

Is there any ordering?

Vertex ordering

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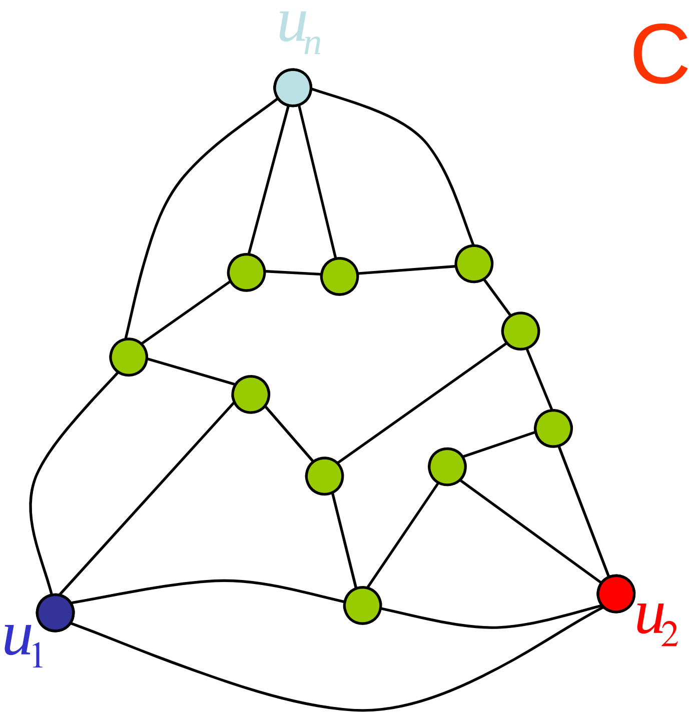
Canonical Decomposition



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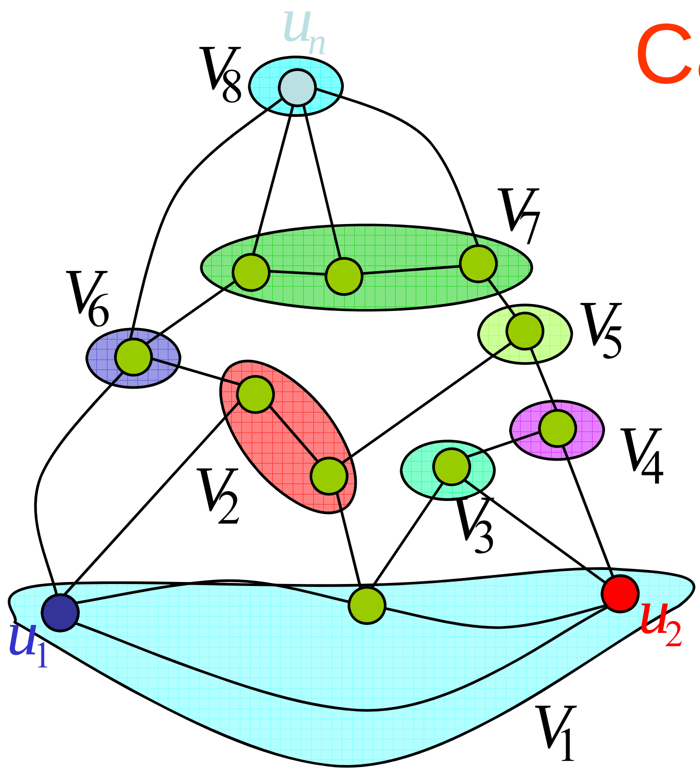
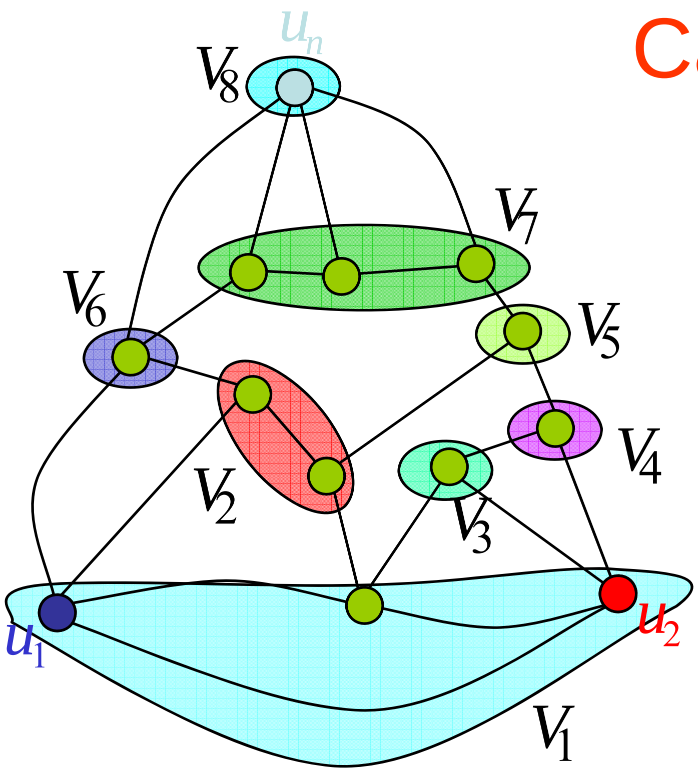




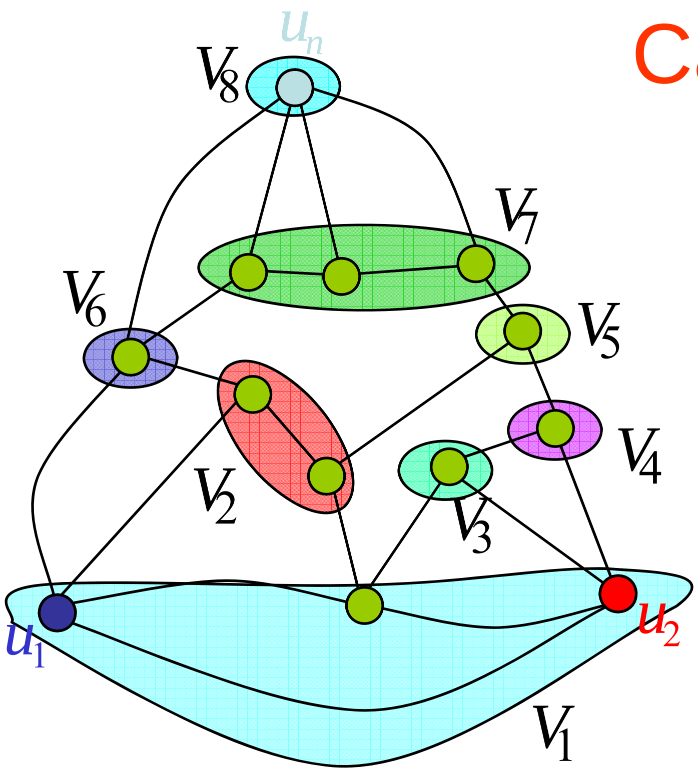
Canonical Decomposition

(cd1) V_1 is the set of all vertices on the inner face containing edge (u_1,u_2).

(cd2) for each index k, $1 \leq k \leq h$, G_k is internally 3-connected.

(cd3) for each k, $2 \leqq k \leqq h\text{-}1$, vertices in V_k are on the outer vertices and the following (a) and (b) holds.

Canonical Decomposition



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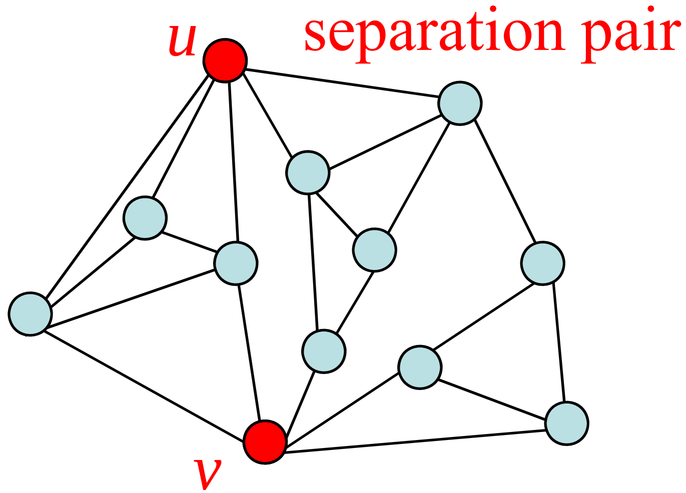
Internally 3-connected

G is bi-connected

For any separation pair $\{u,v\}$ of G

u and v are outer vertices

each connected component of G-$\{u,v\}$ contains an outer vertex.



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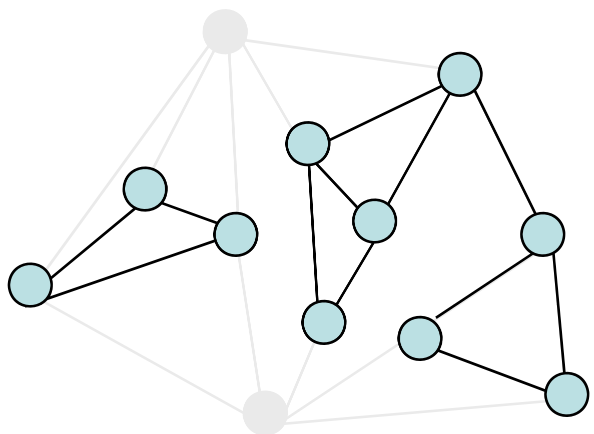

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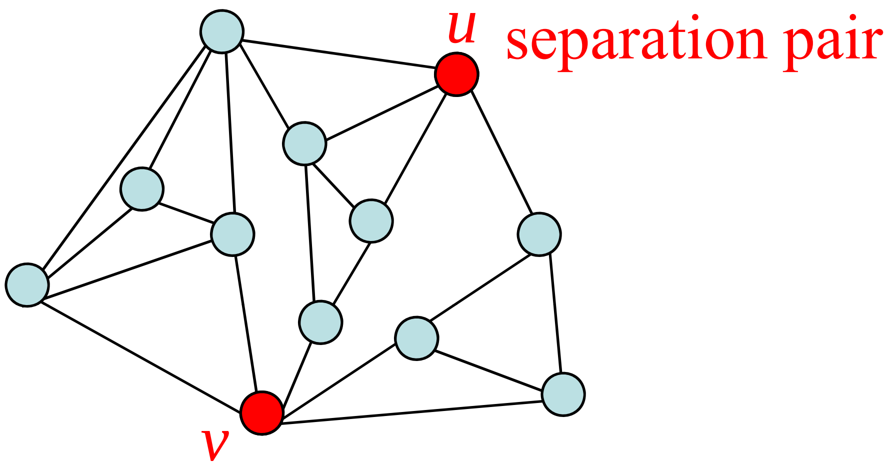
v_2



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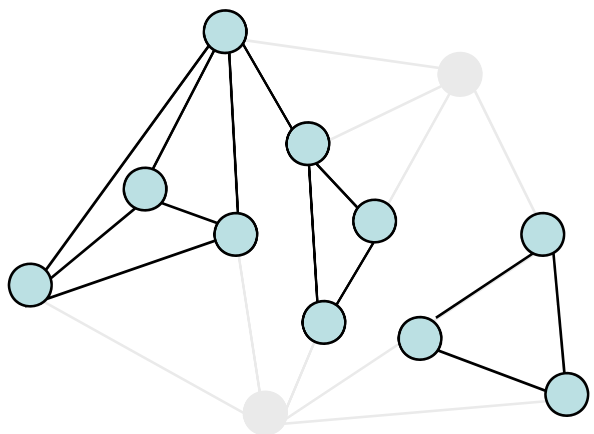

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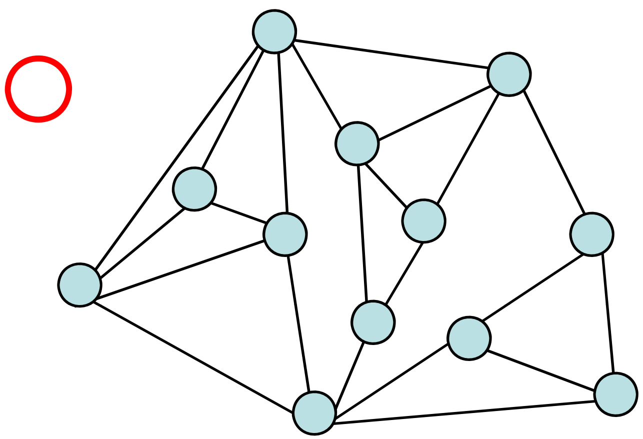

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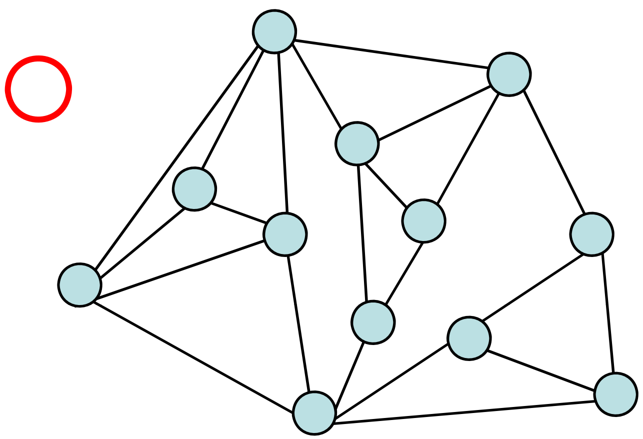

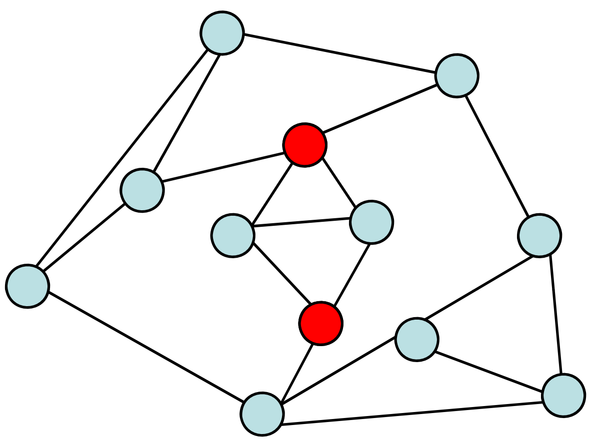

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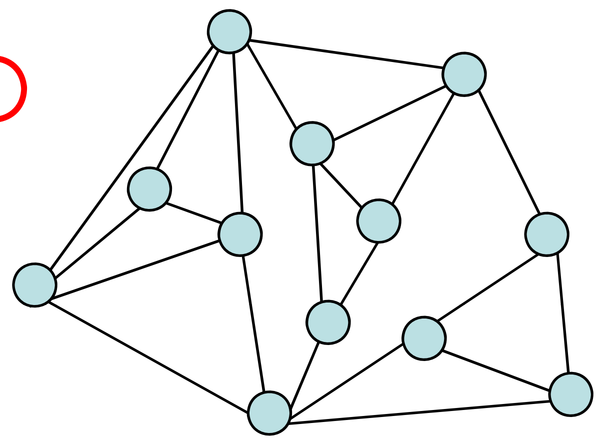

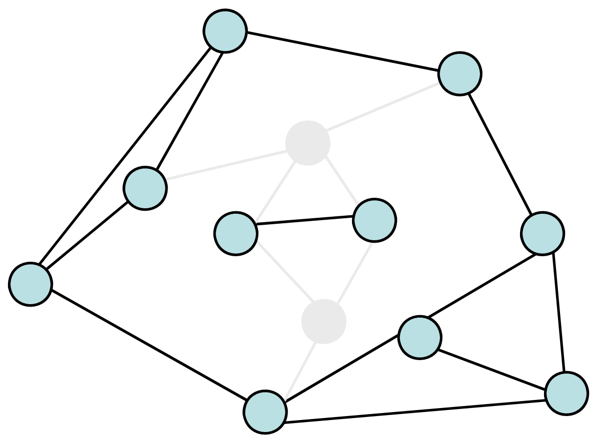

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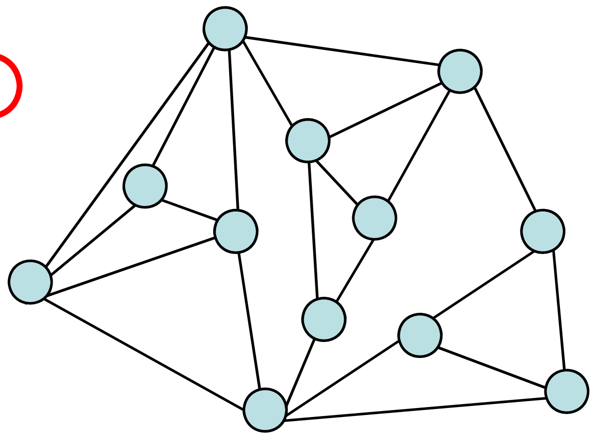

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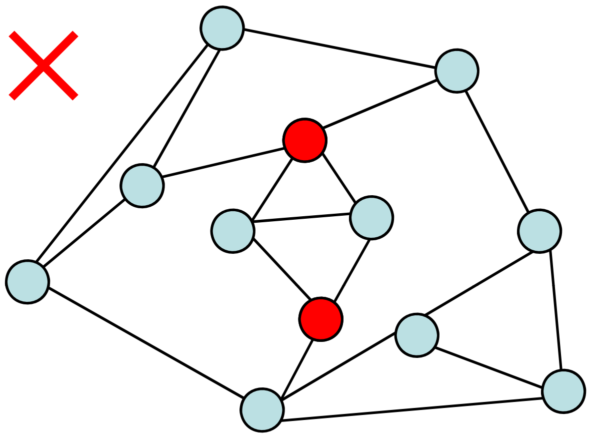

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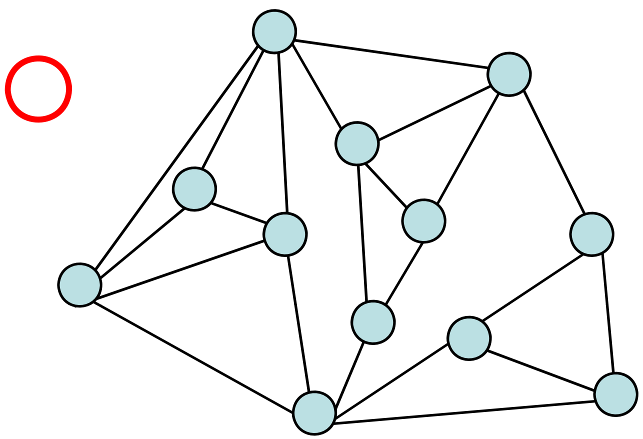

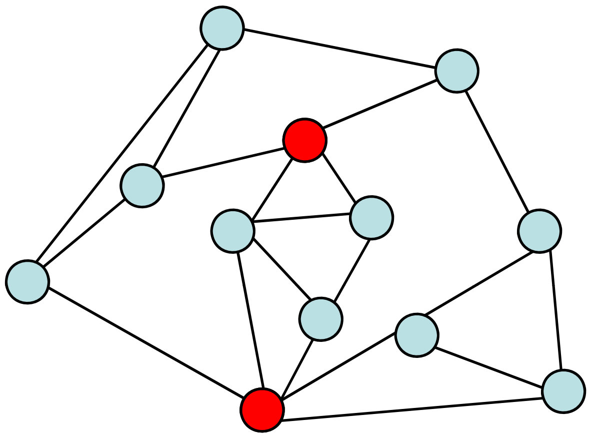

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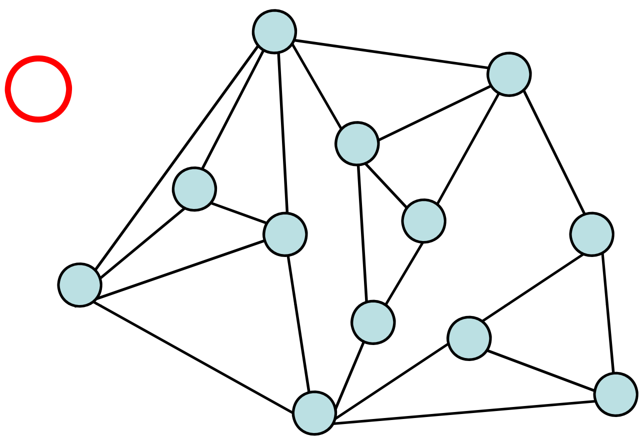

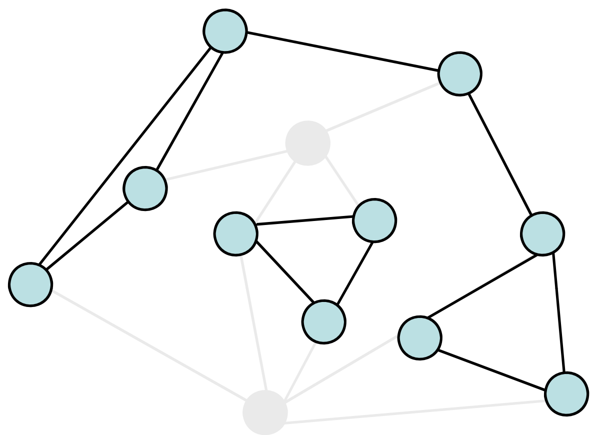

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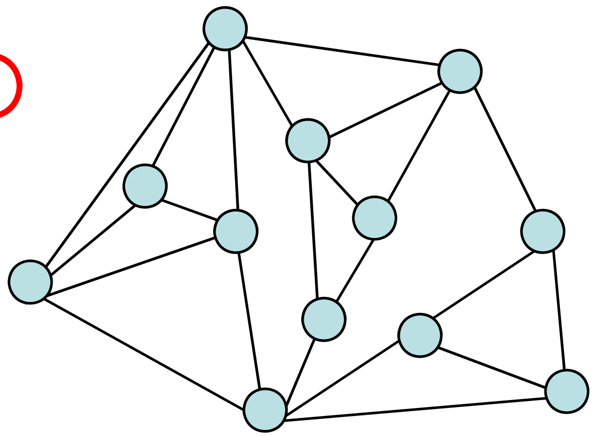

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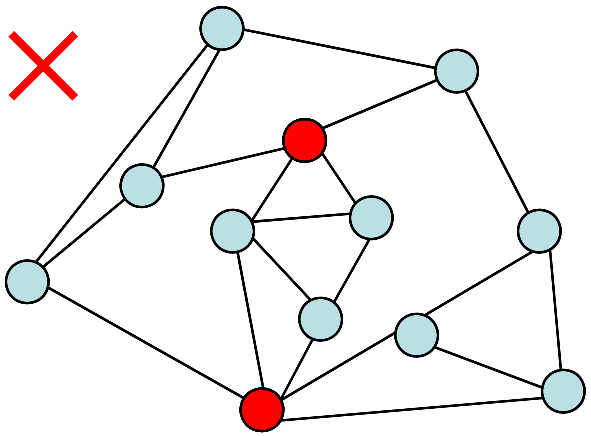

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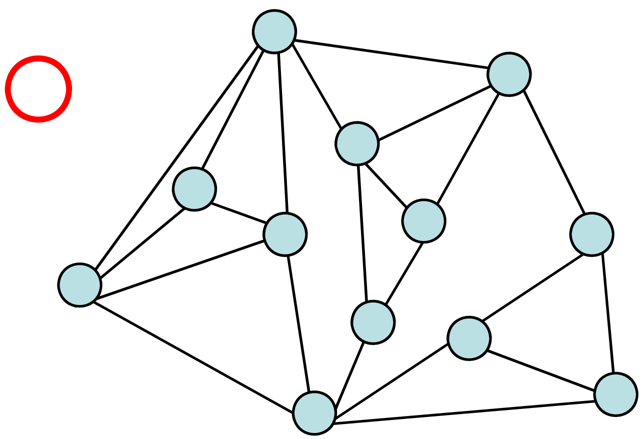

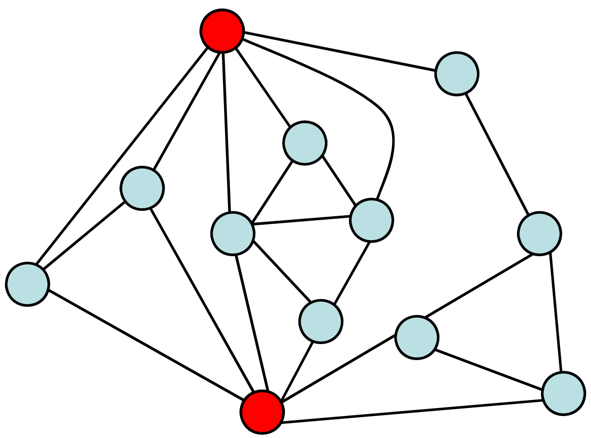

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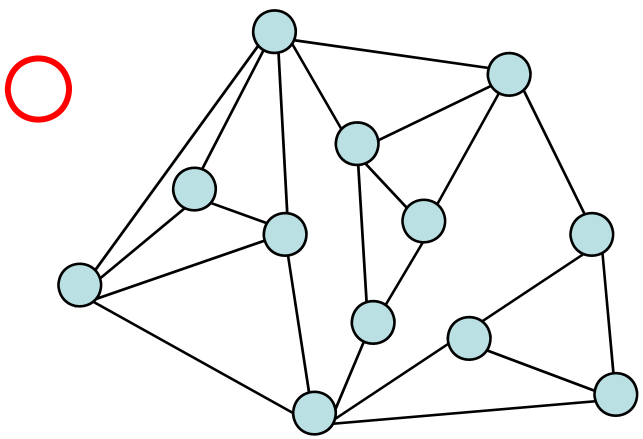
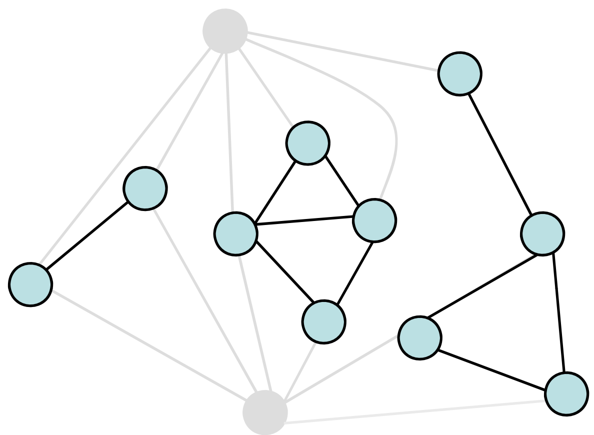




Internally 3-connected

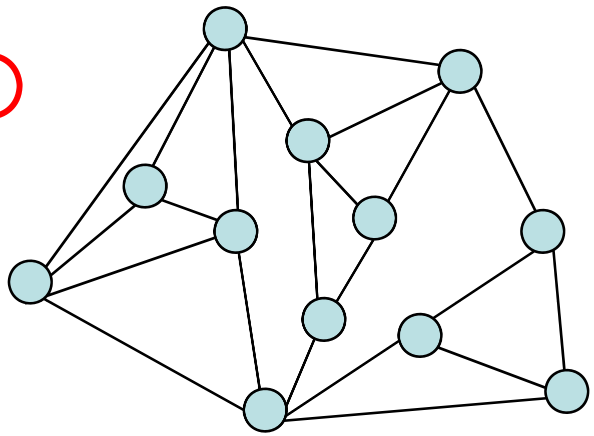
G is bi-connected

For any separation pair {*u*,*v*} of *G*

u and *v* are outer vertices

each connected component of *G*-{*u*,*v*} contains an outer vertex.

○



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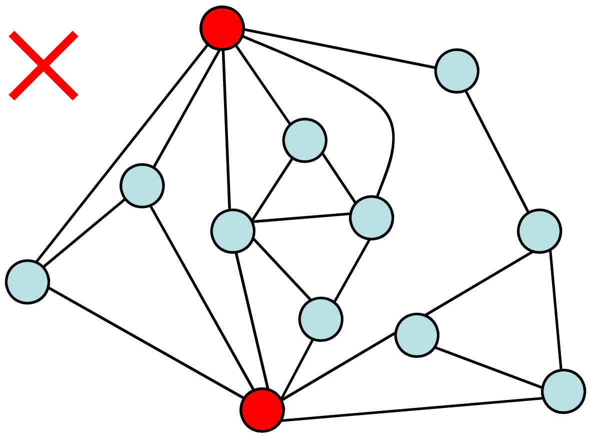
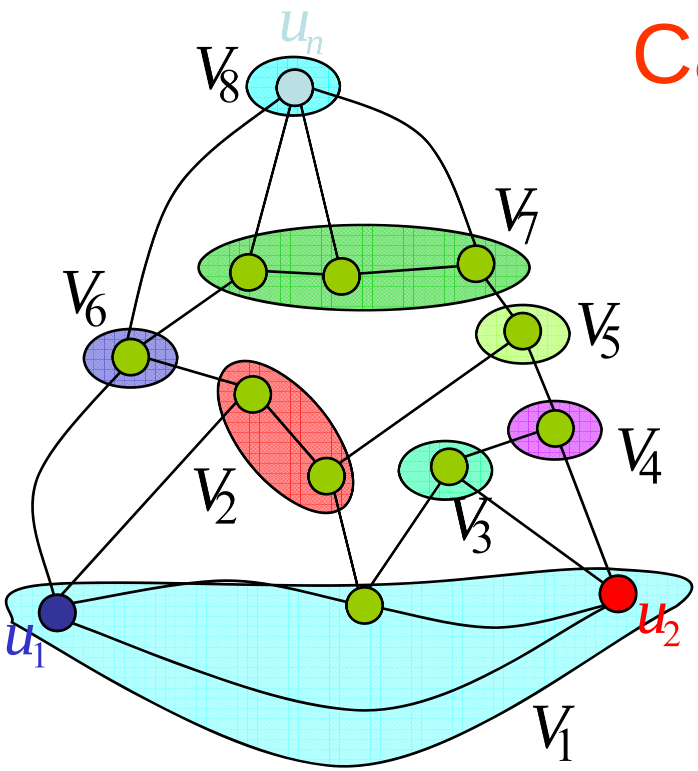




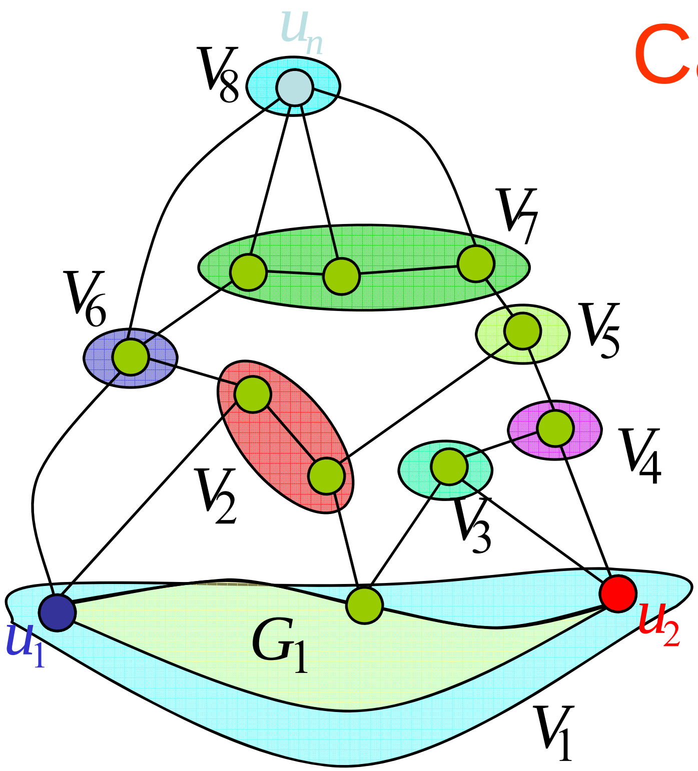
Canonical Decomposition

(cd1) V_1 is the set of all vertices on the inner face containing edge (u_1,u_2).

(cd2) for each index k, $1 \leq k \leq h$, G_k is internally 3-connected.

(cd3) for each k, $2 \leqq k \leqq h-1$, vertices in V_k are on the outer vertices and the following (a) and (b) holds.

Canonical Decomposition



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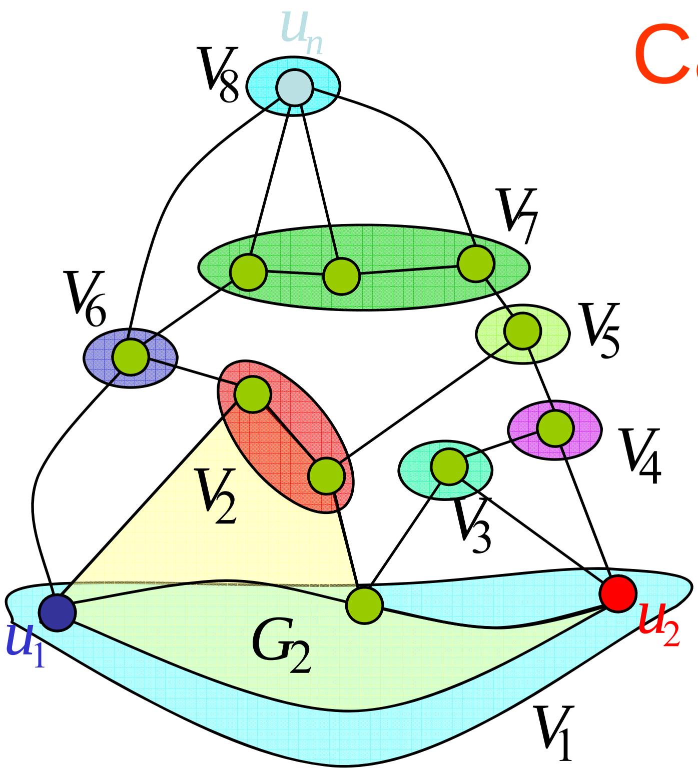

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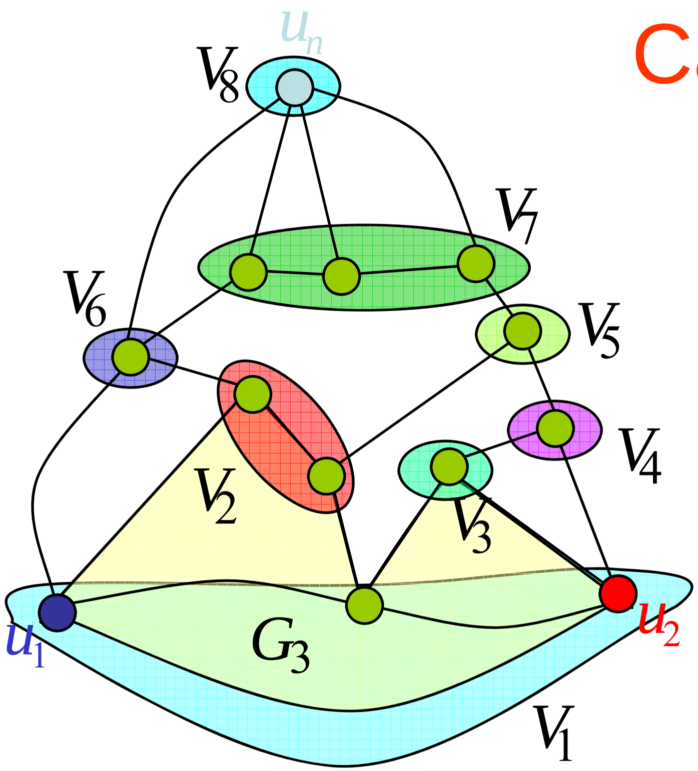

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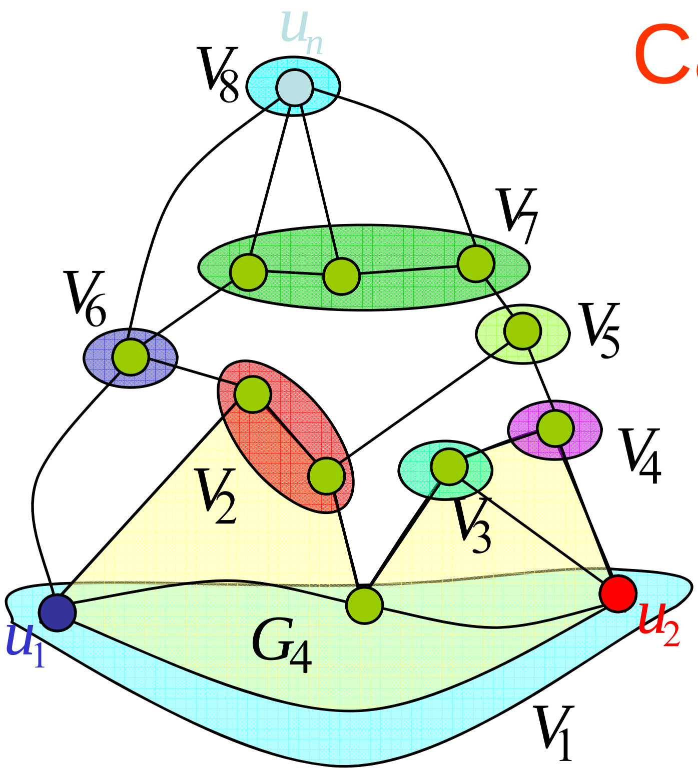

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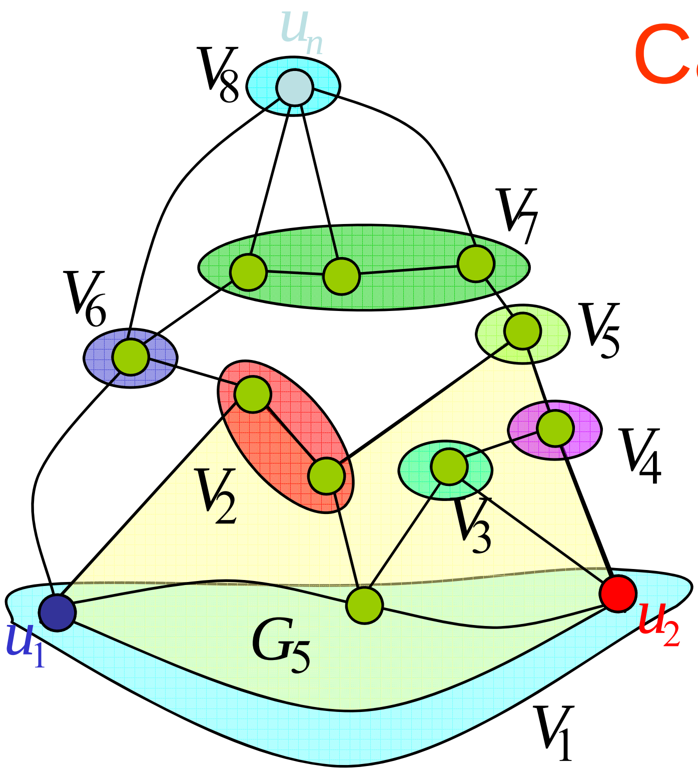

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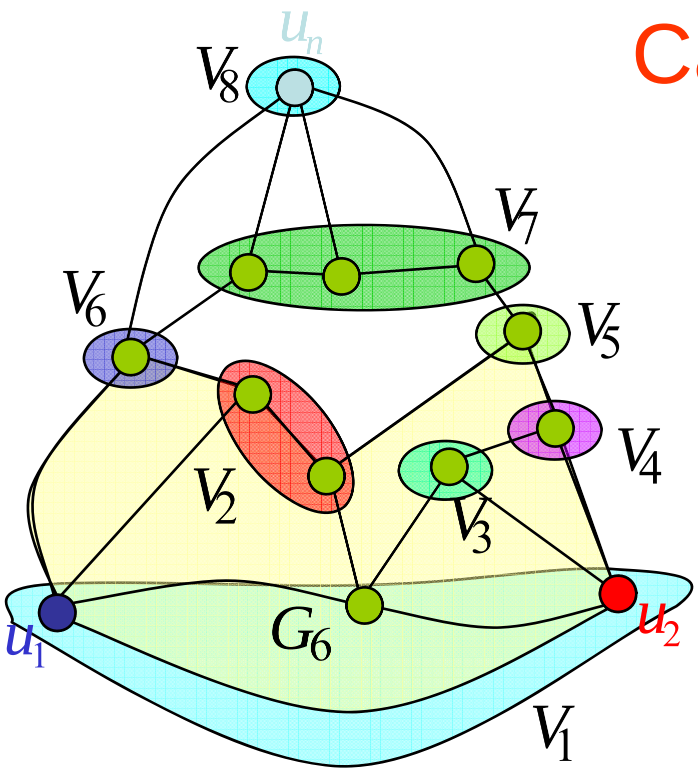

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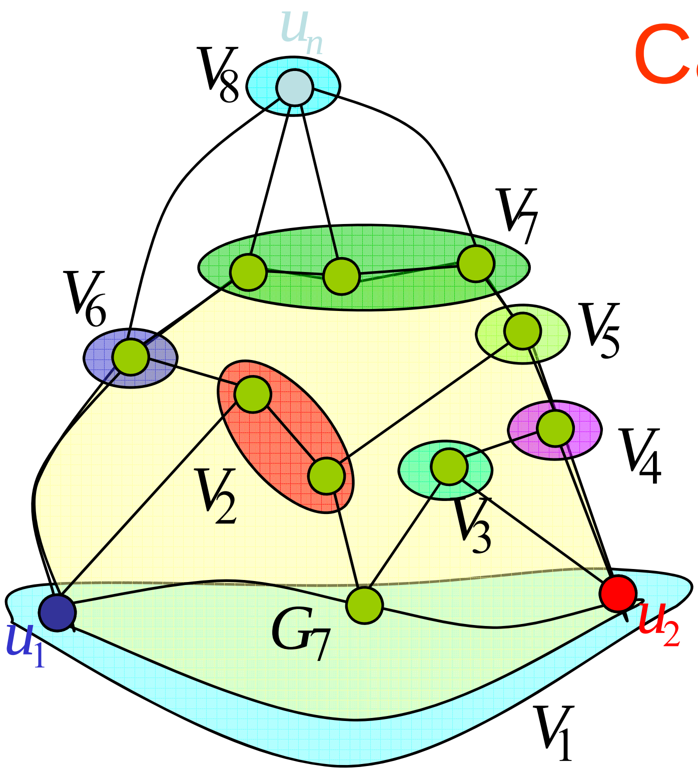

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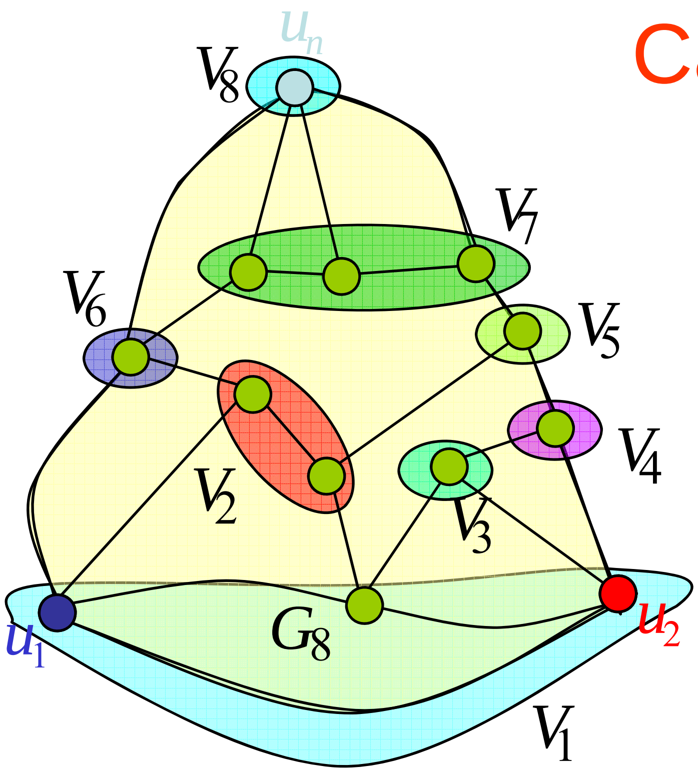

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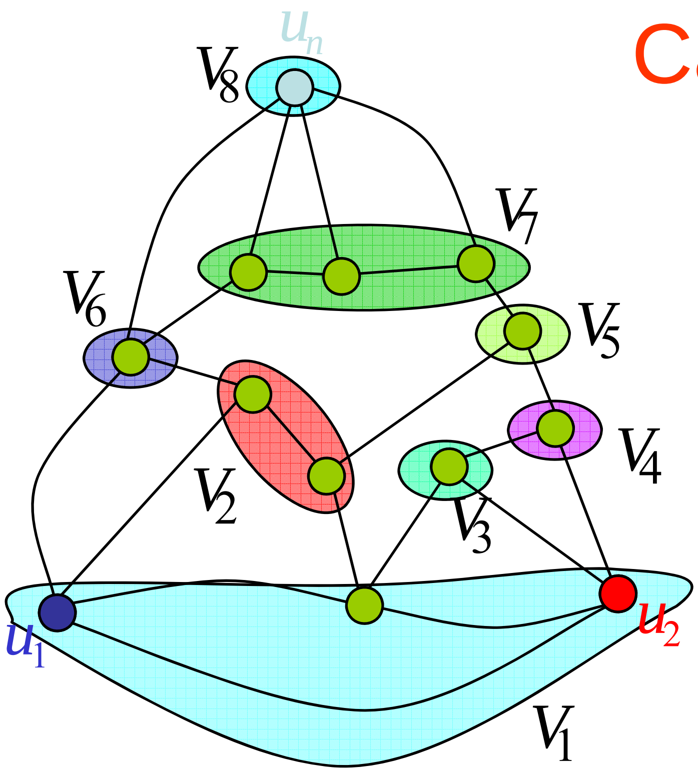

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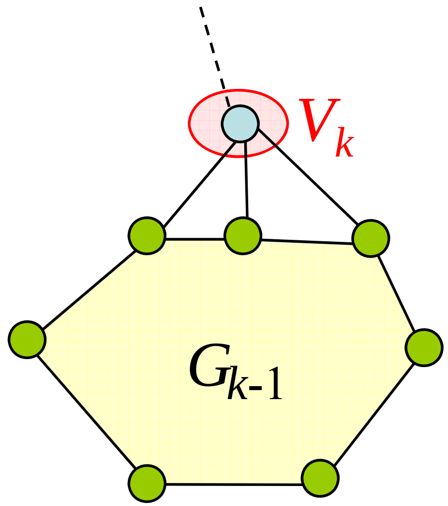
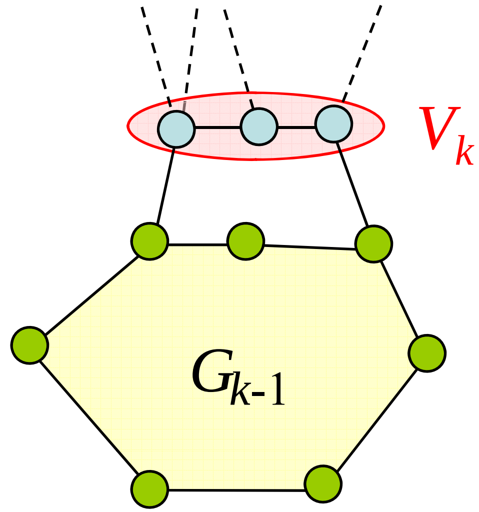
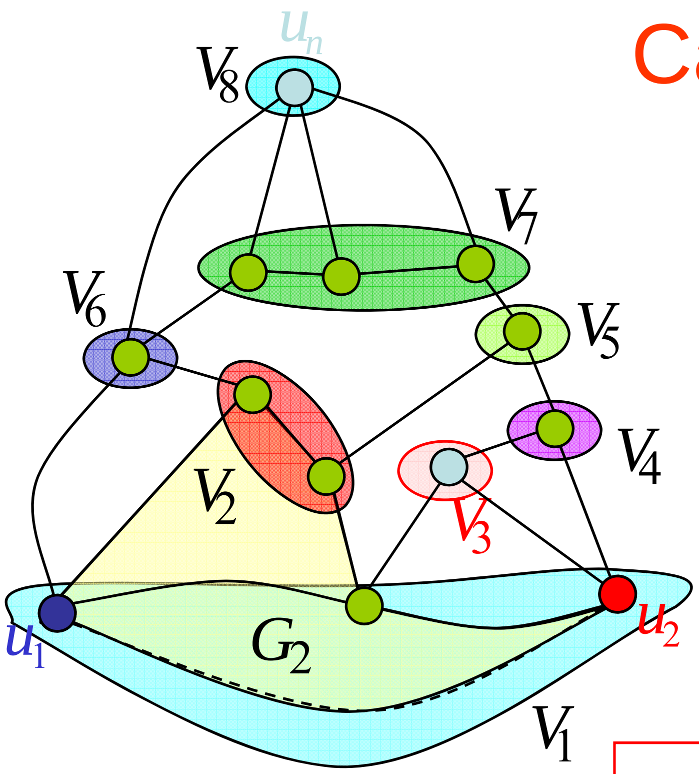


(a)



(b)



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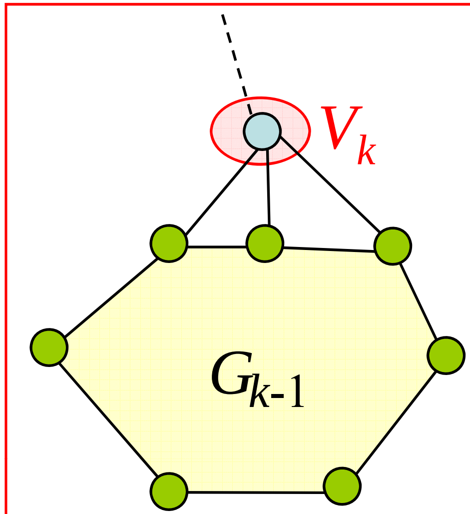
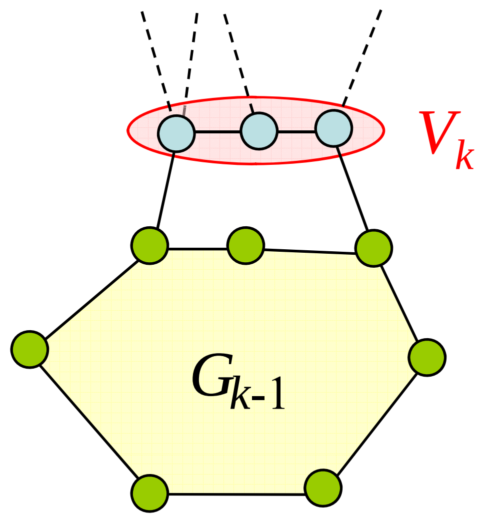
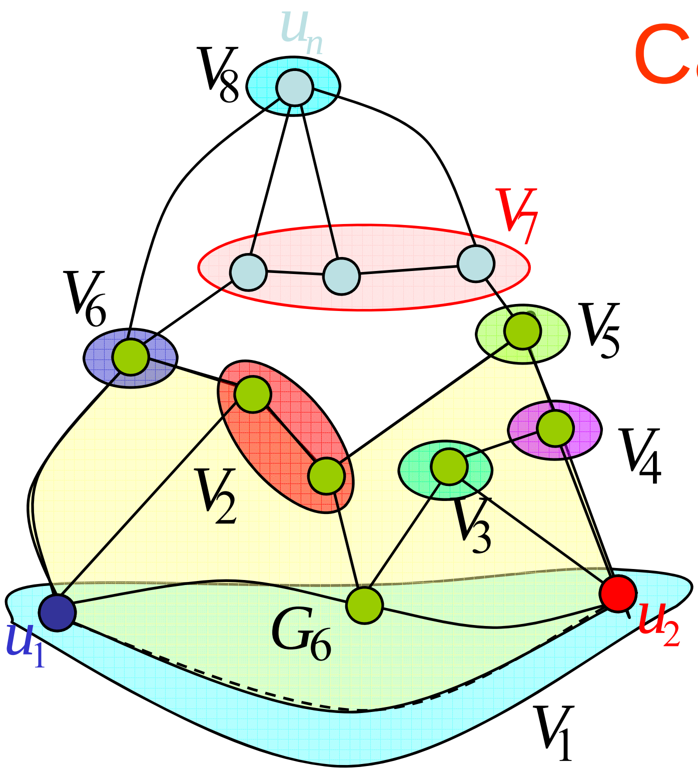


(a)



(b)

Canonical Decomposition



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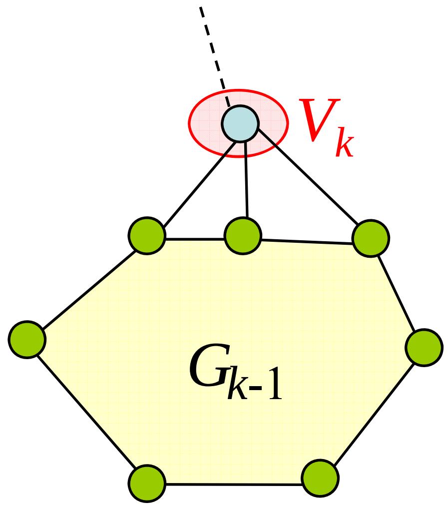
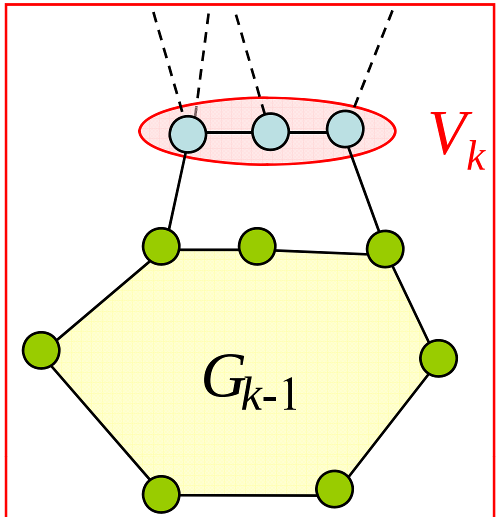


(a)



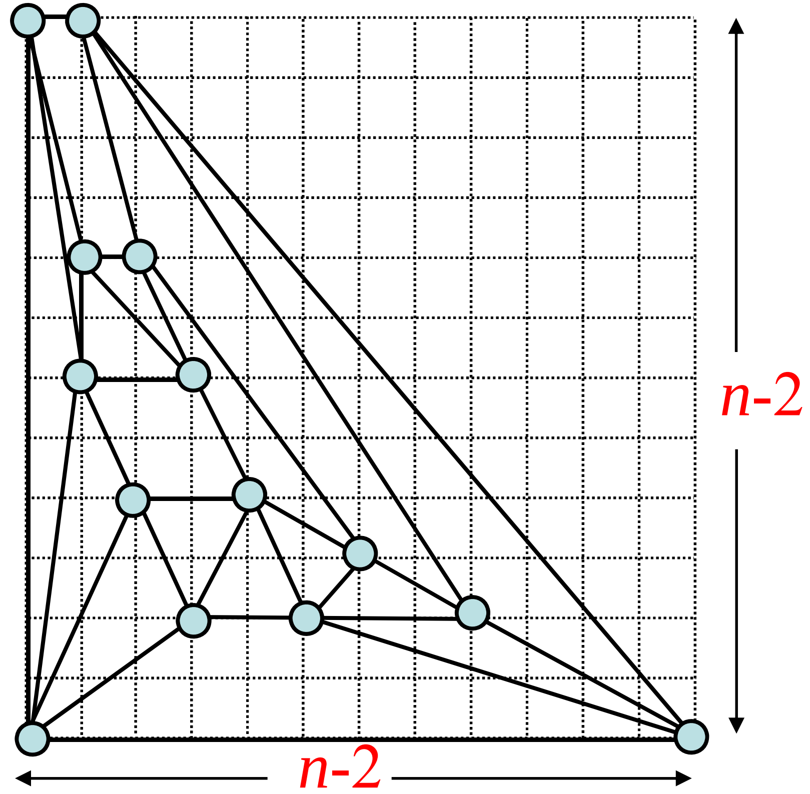
(b)

Convex Grid Drawing

Chrobak and Kant '97

Input: 3-connected graph

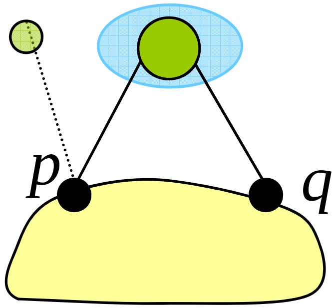
Output: convex grid drawing

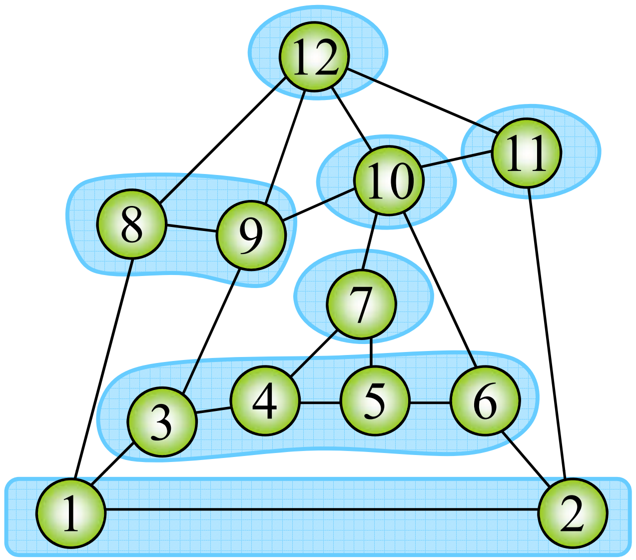


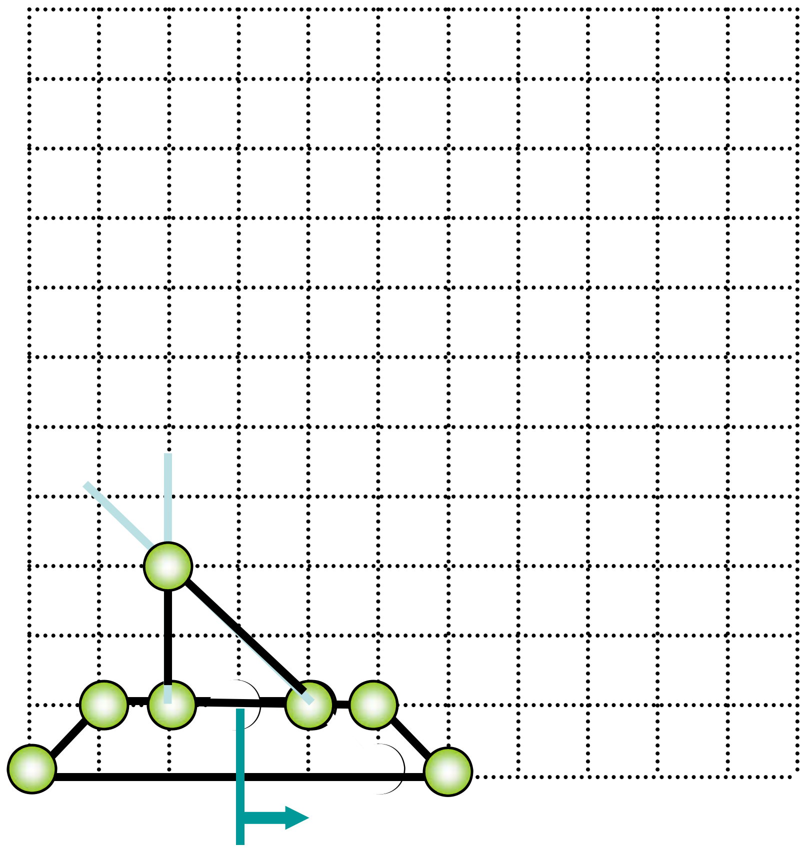
Grid Size

Area $W \times H \leq n^2$

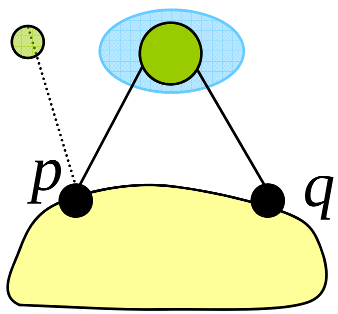
Shift method

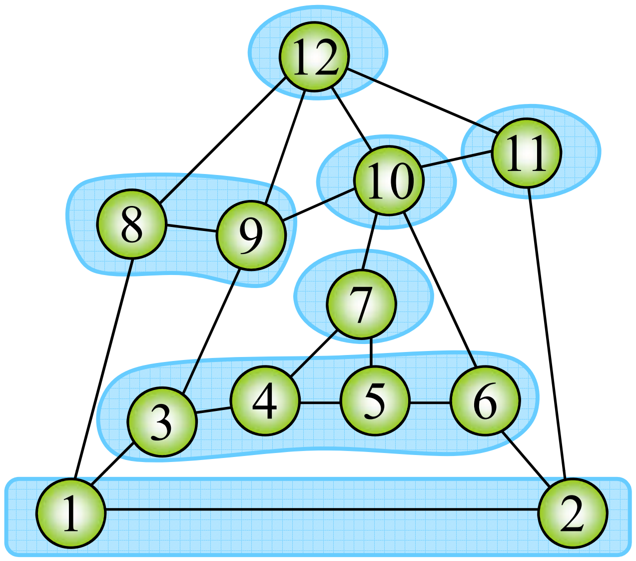


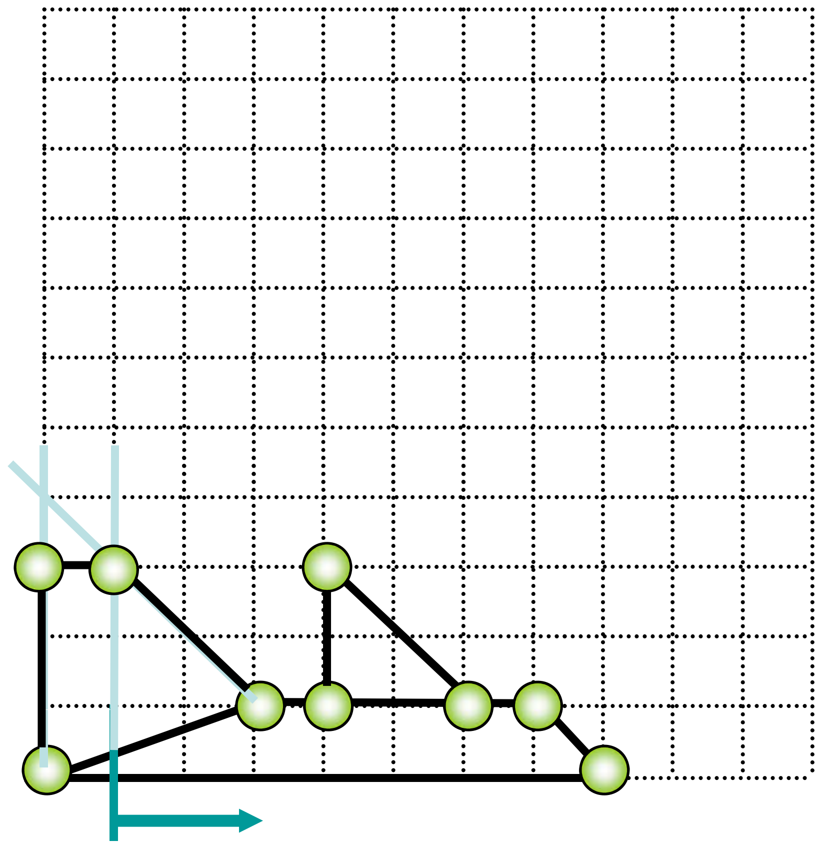




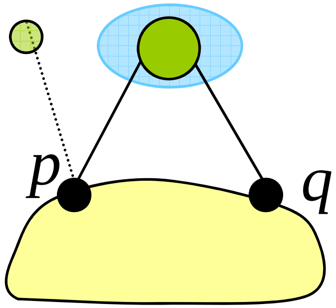
Shift method

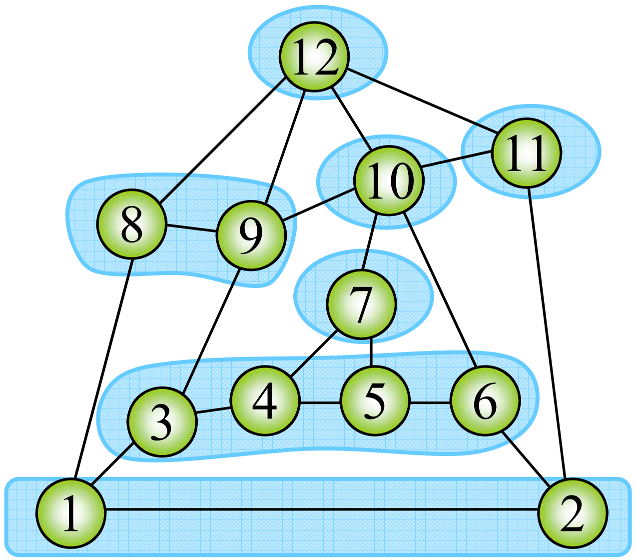


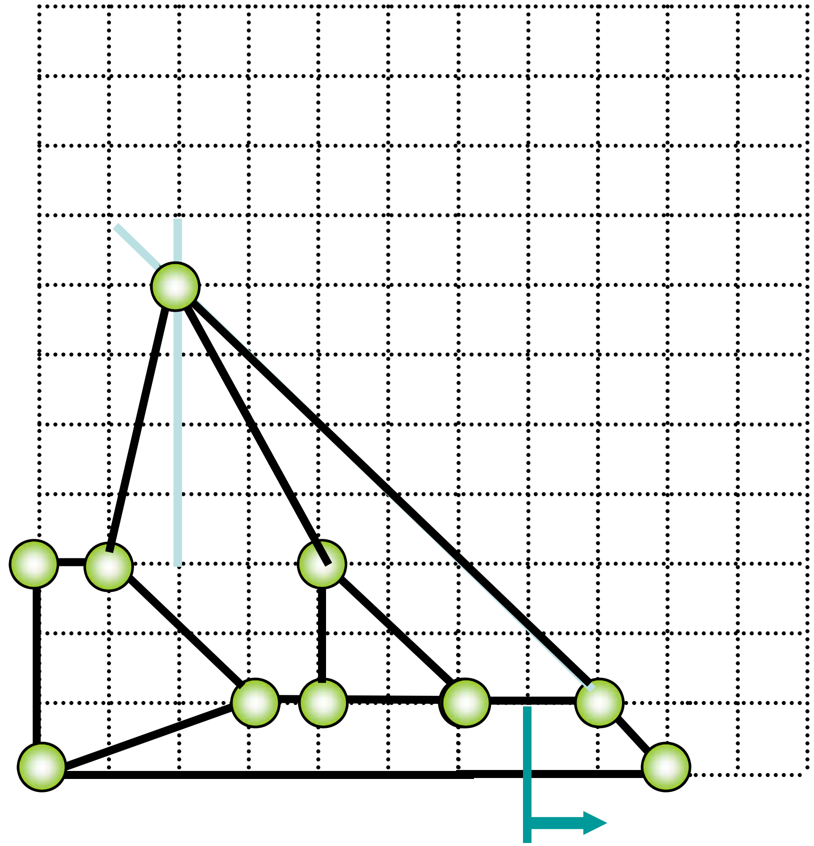




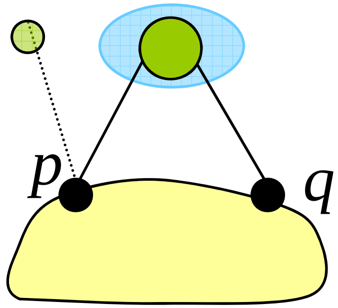
Shift method

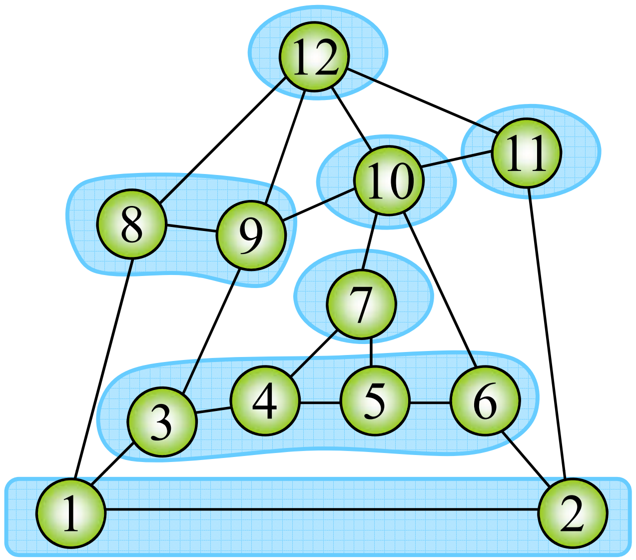


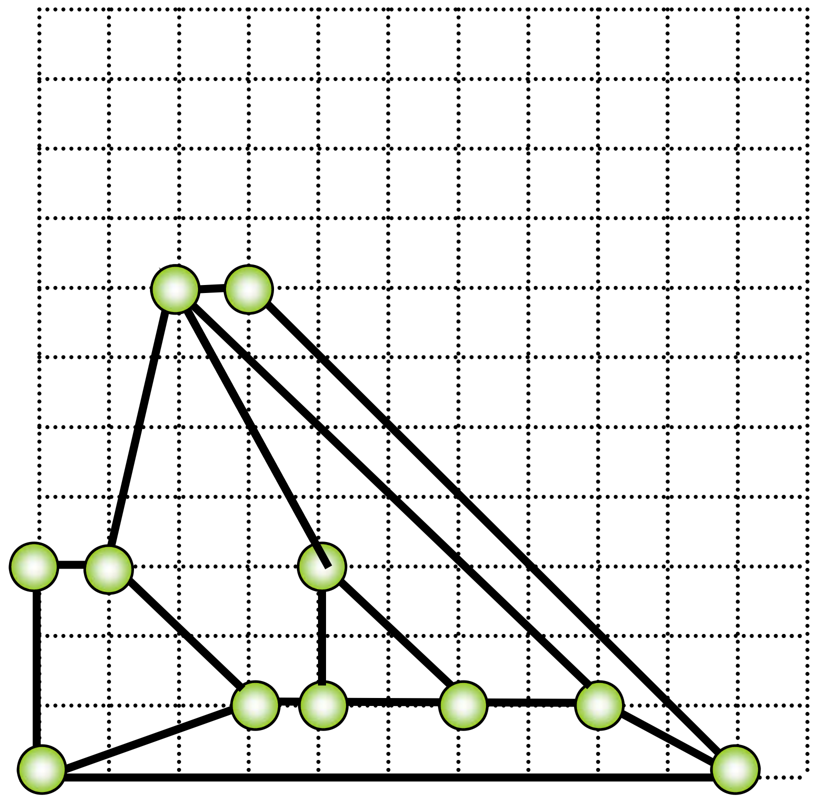




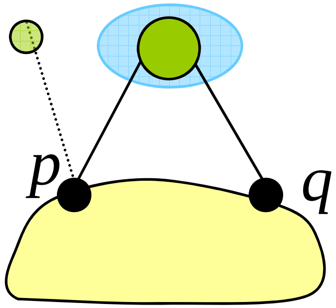
Shift method

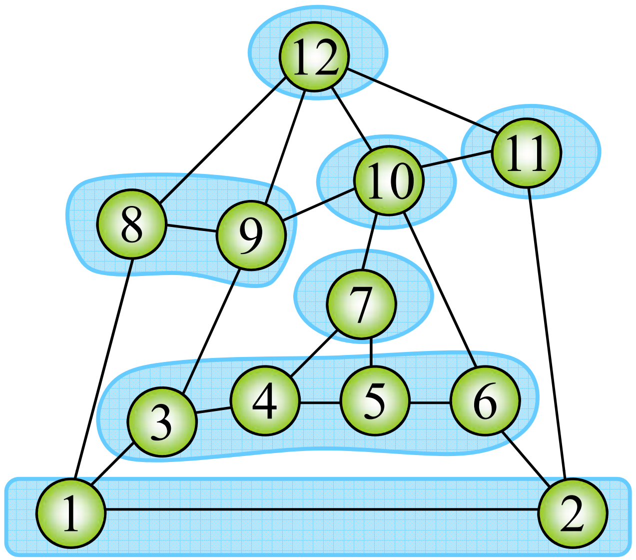


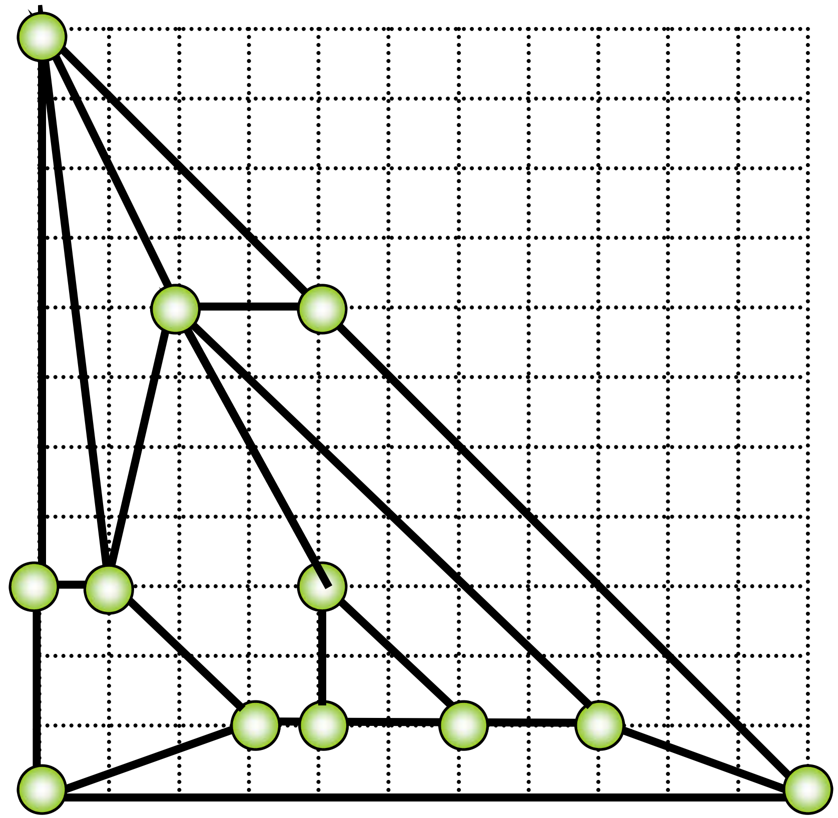




Shift method

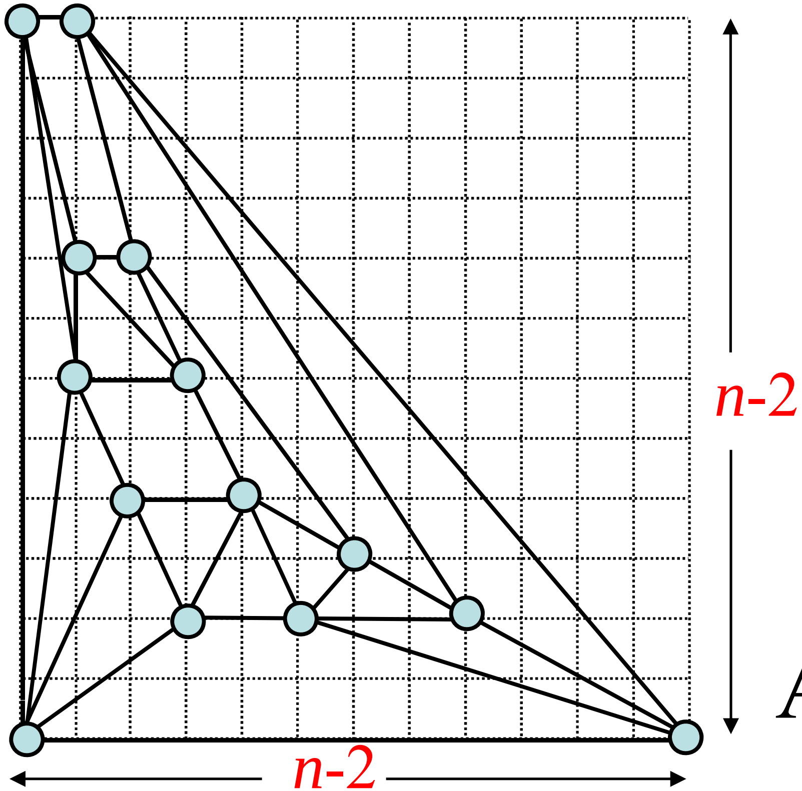






Chrobak and Kant '97

3-connected graph

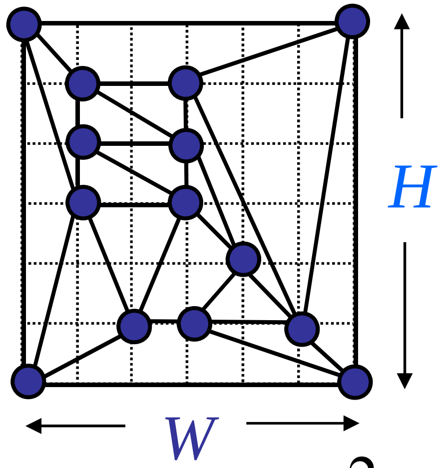


Area $\approx n^2$

Miura *et al.* 2000

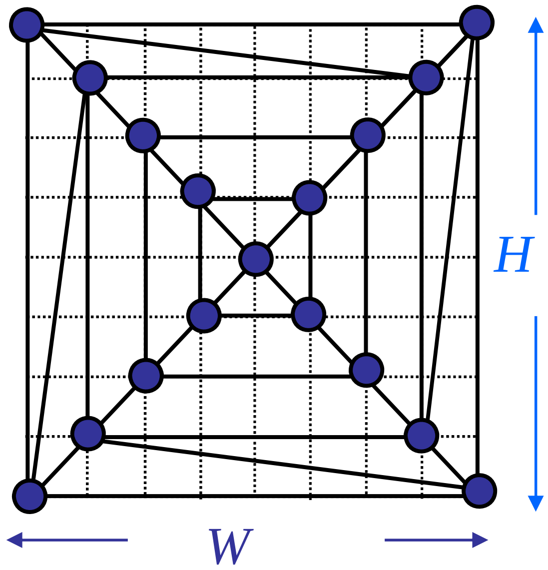
4-connected graph

Area $\leqq 1/4$



Area $\leq \frac{n^2}{4}$

The algorithm of Miura *et al*. is
best possible

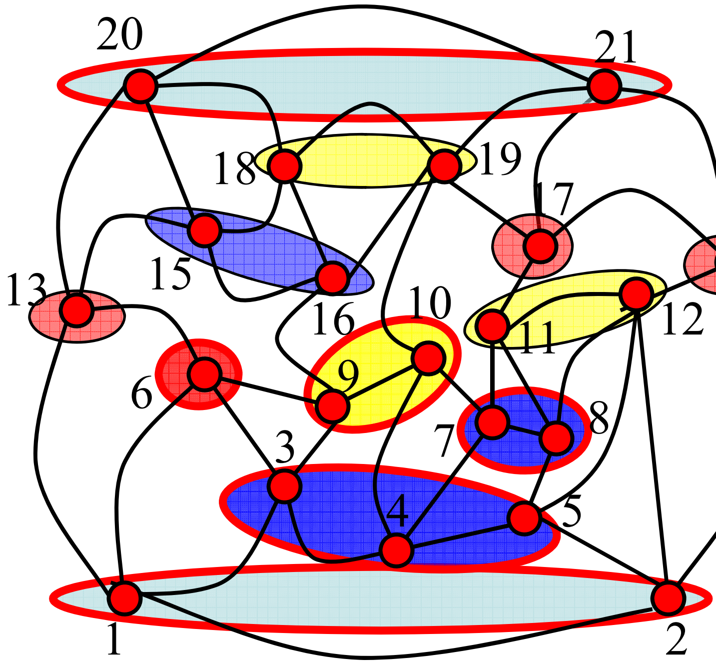


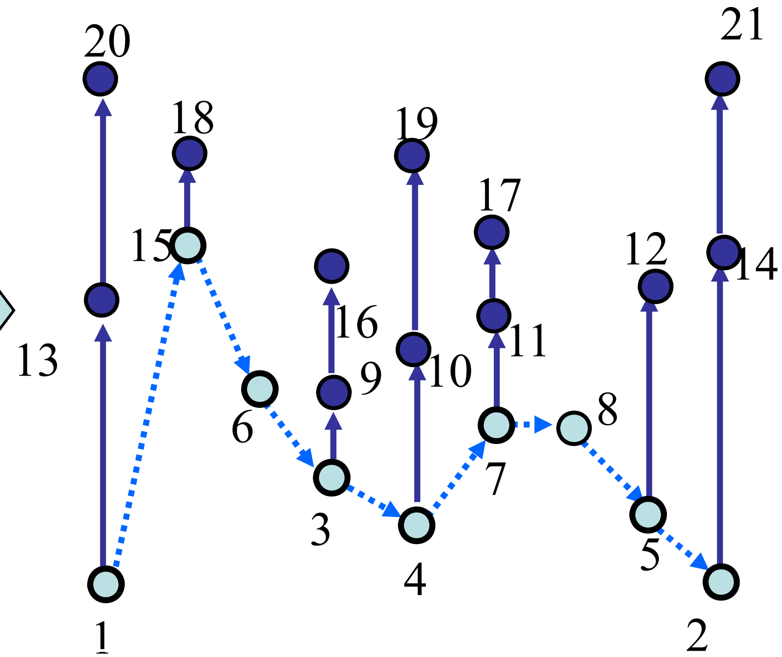
$$W \times H \geq \frac{n^2}{4}$$

Vertex ordering

- *st*-numbering
- Canonical ordering
- 4-canonical ordering
- Canonical decomposition
- 4-canonical decomposition

Main idea



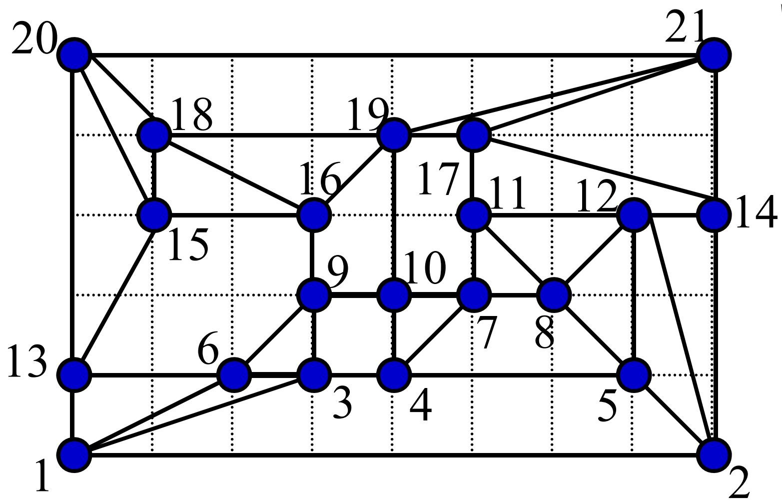


1: 4-canonical decomposition

$O(n)$[NRN97]

2: Find paths

3: Decide x-coordinates

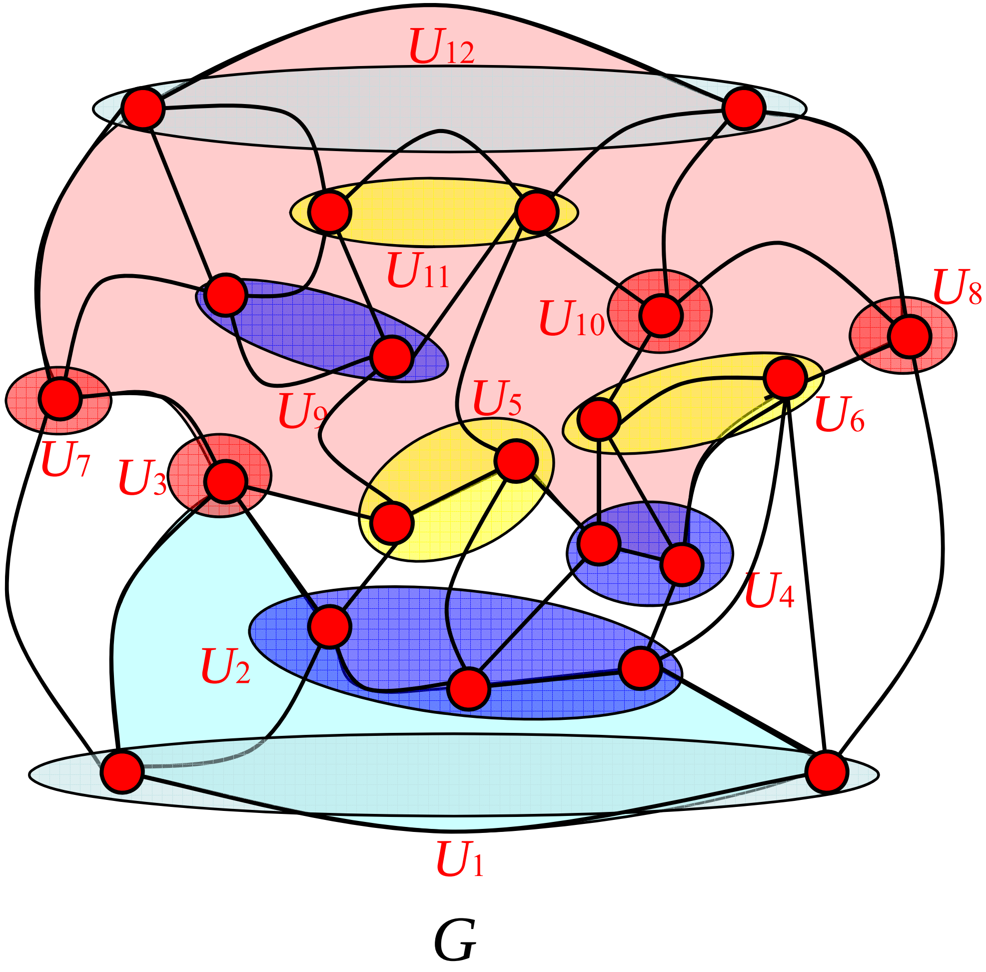


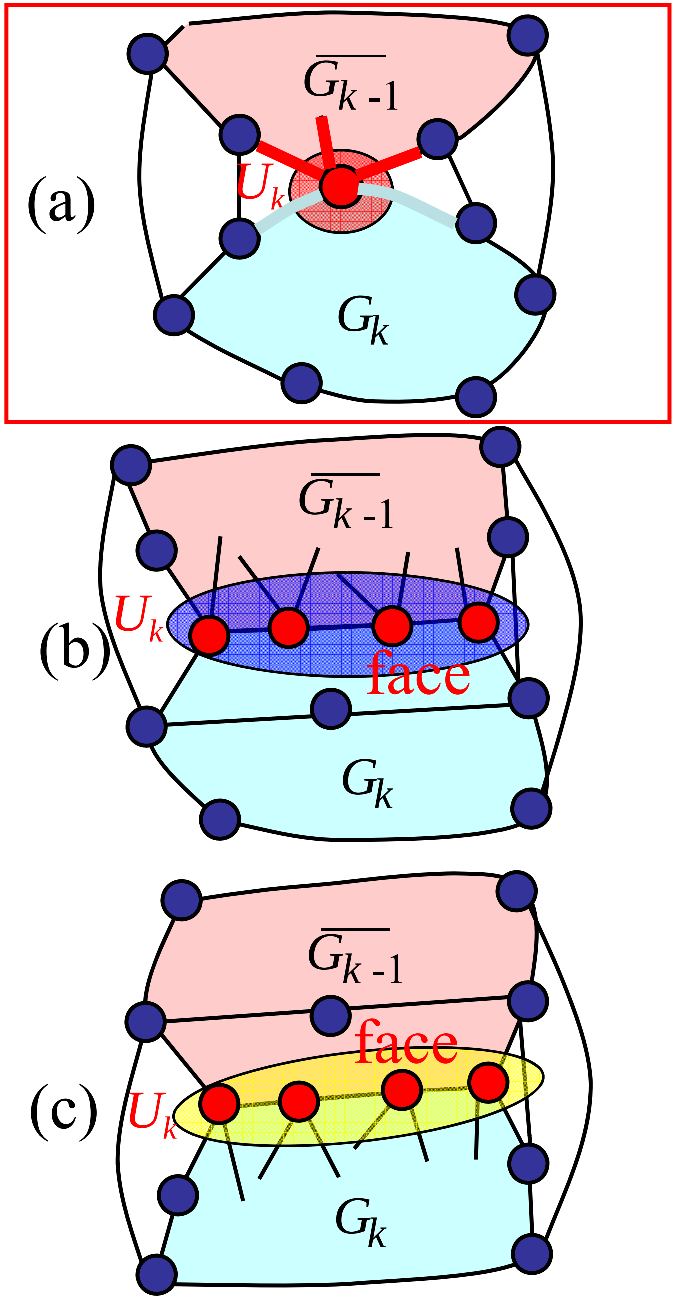
4: Decide y-coordinates

Time complexity: $O(n)$

4-canonical decomposition[NRN97]

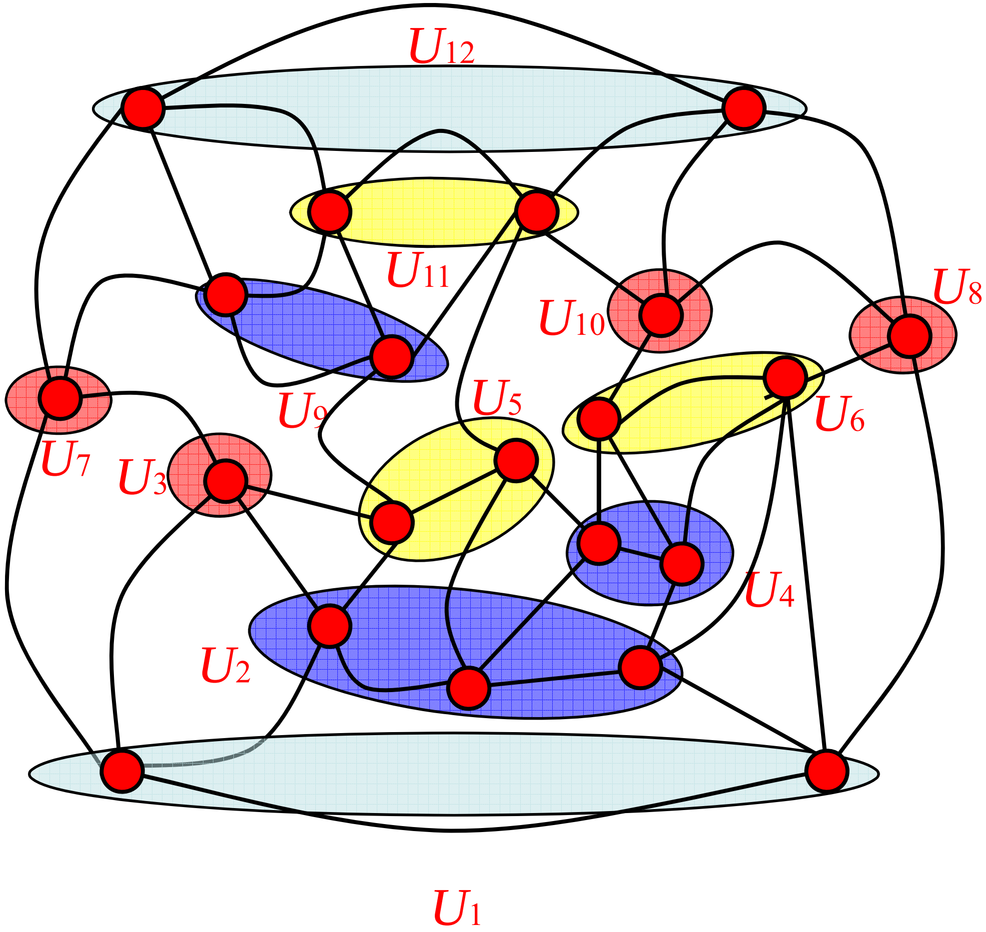
(a generalization of *st*-numbering)

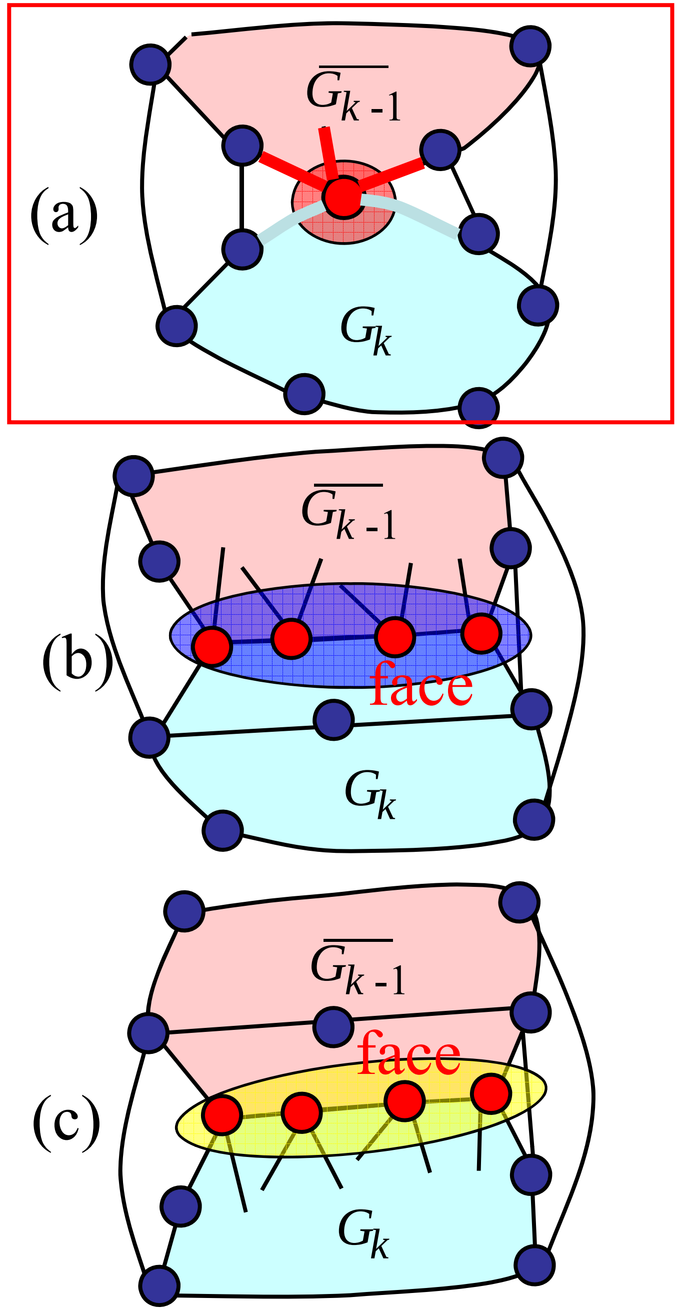




4-canonical decomposition[NRN97]

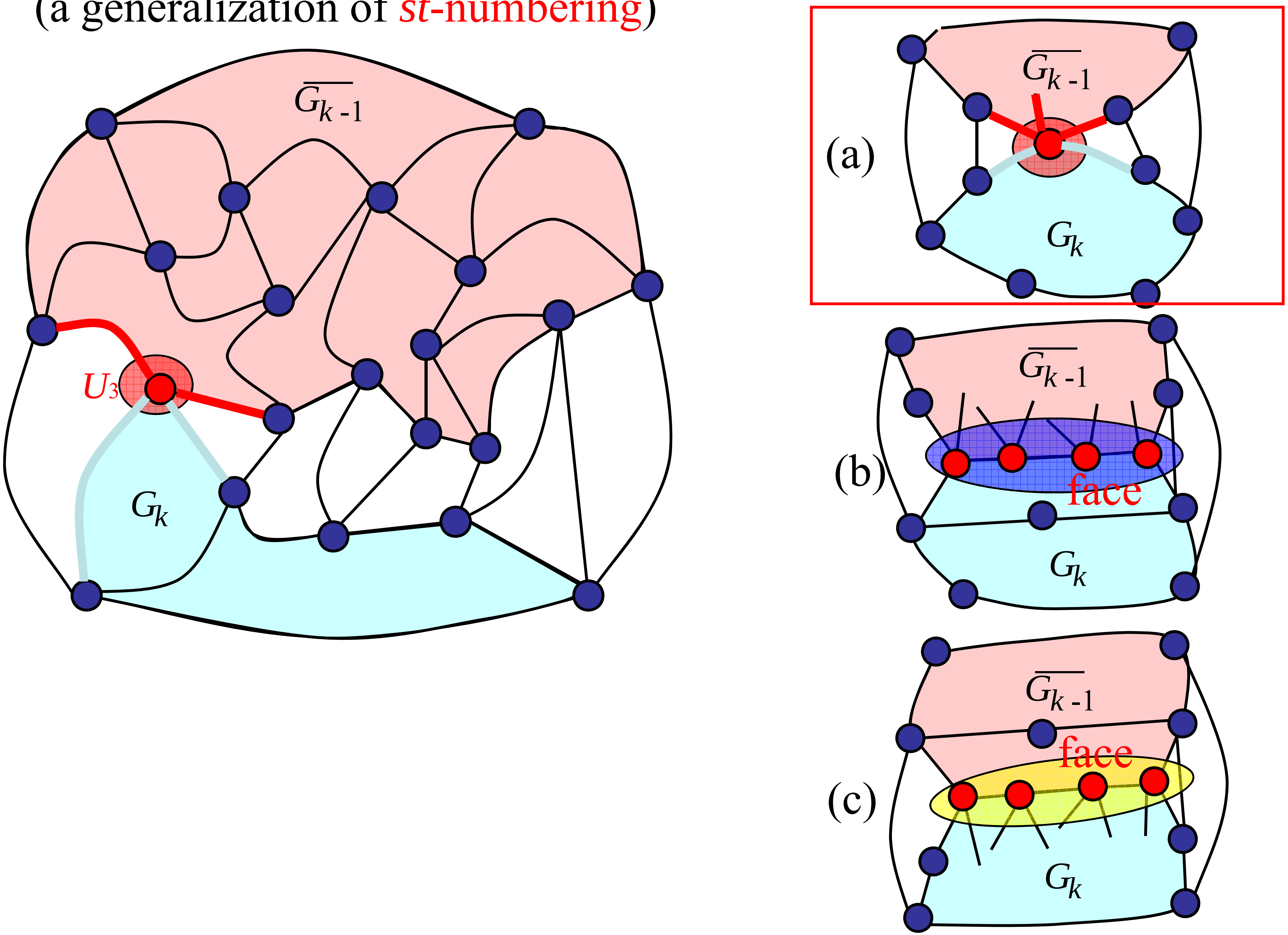
(a generalization of *st*-numbering)





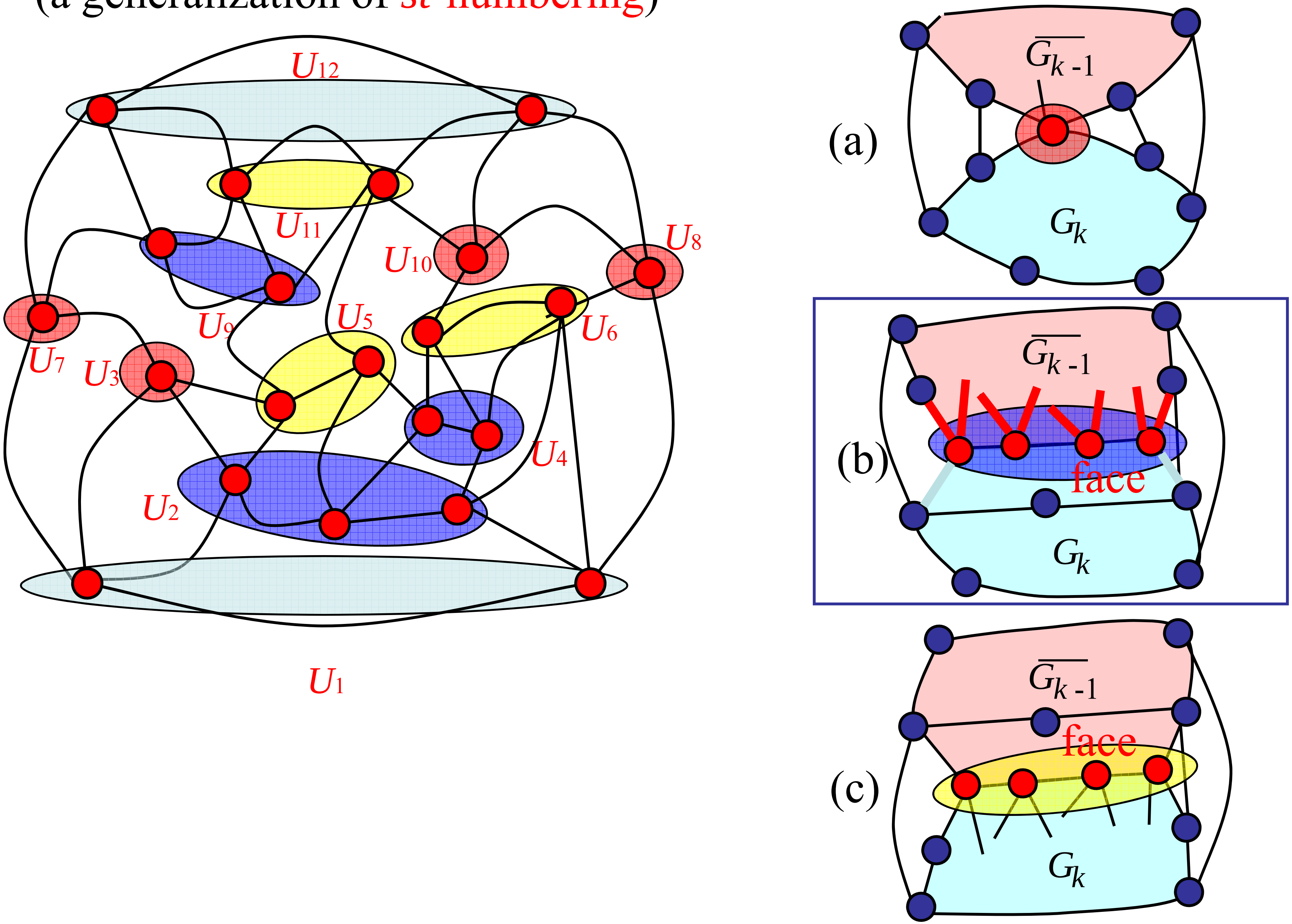
4-canonical decomposition[NRN97]

(a generalization of *st*-numbering)



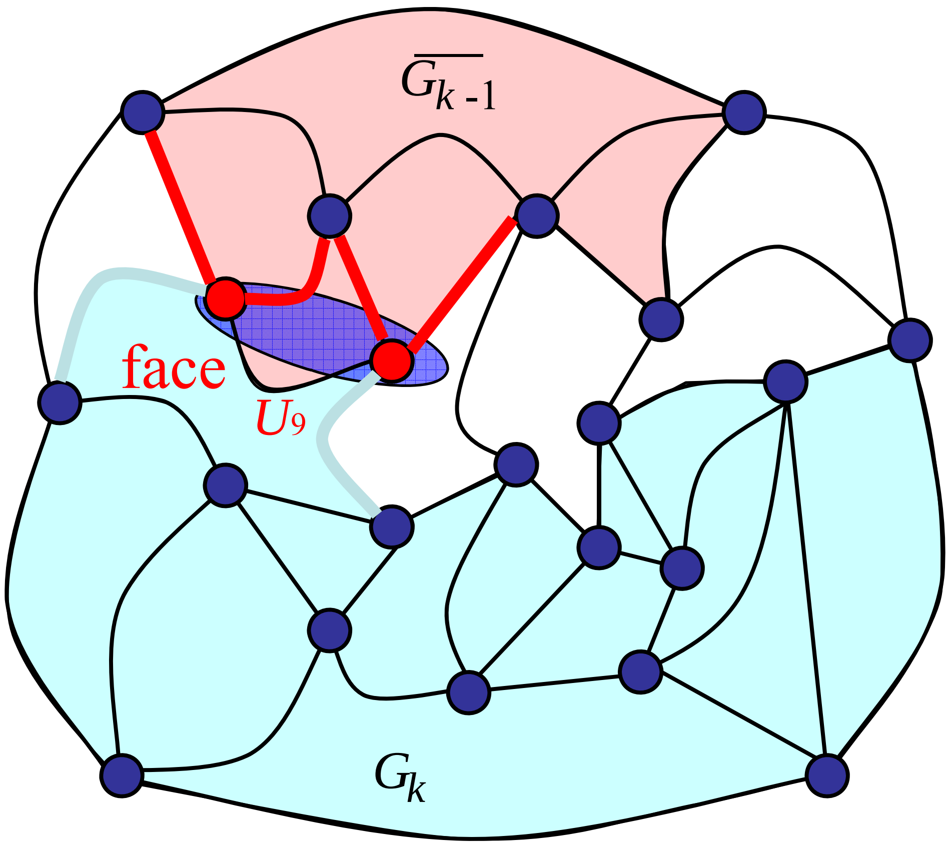
4-canonical decomposition[NRN97]

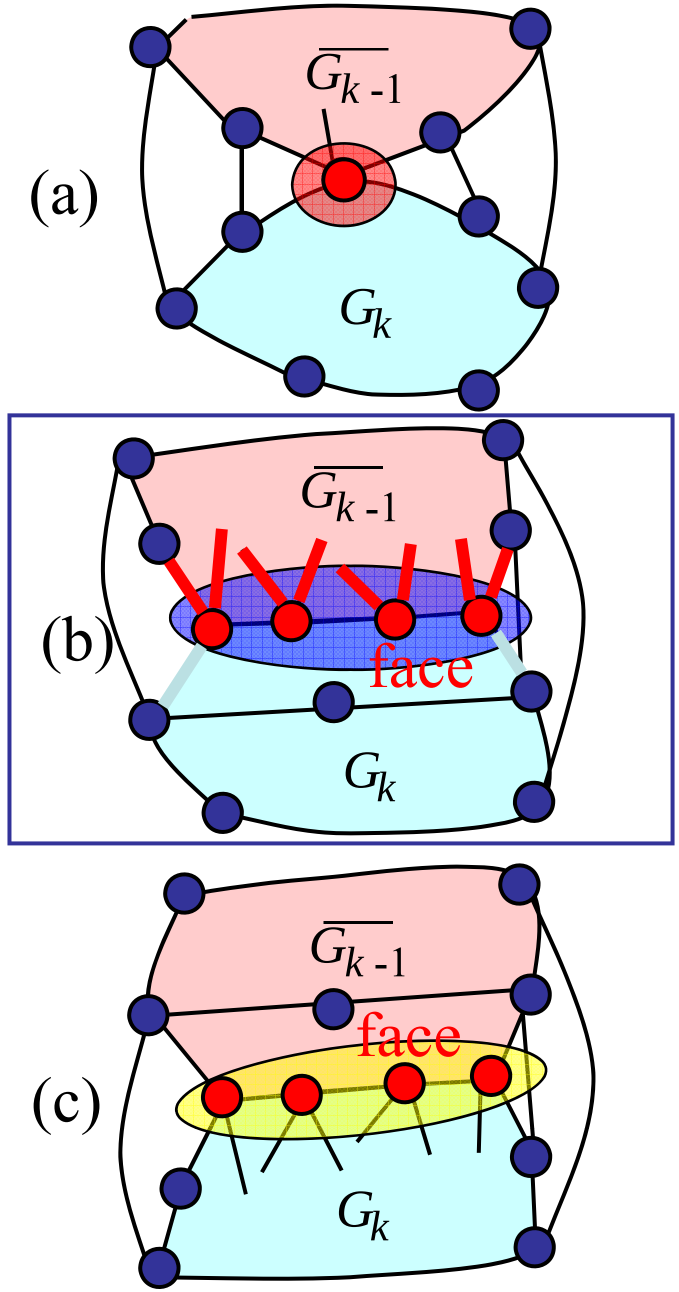
(a generalization of *st*-numbering)



4-canonical decomposition[NRN97]

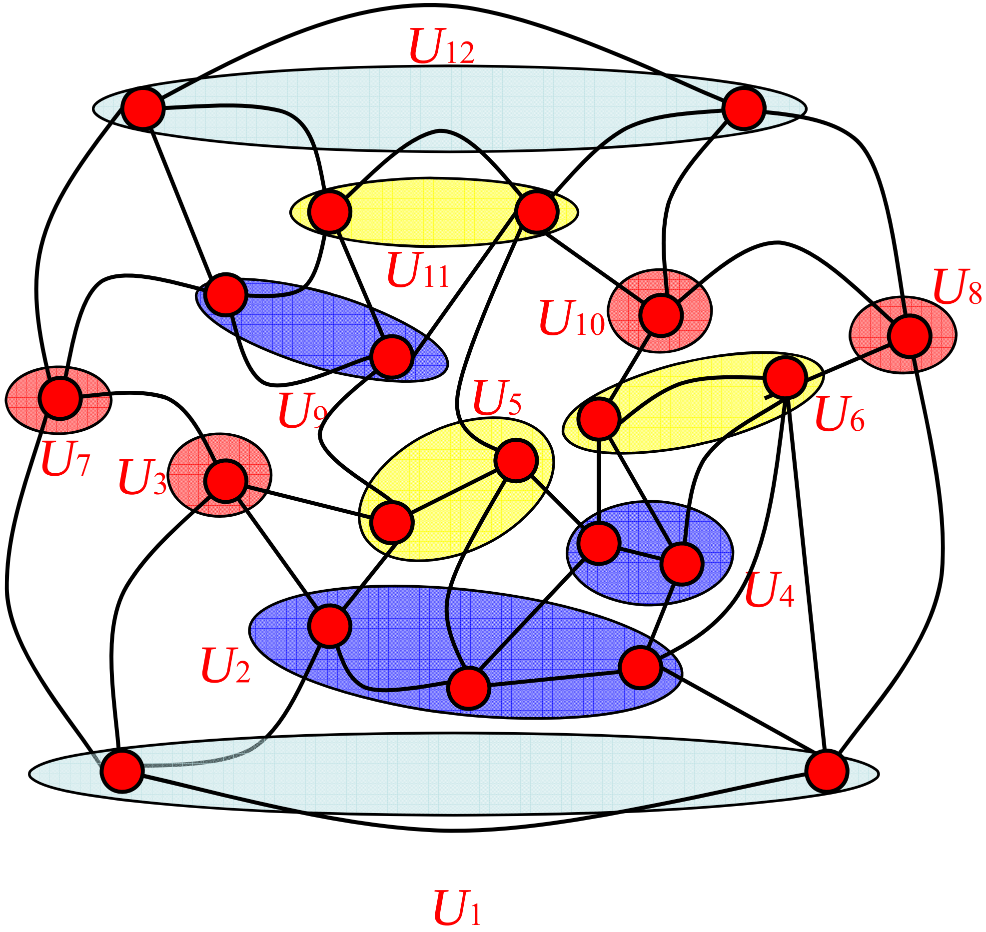
(a generalization of *st*-numbering)

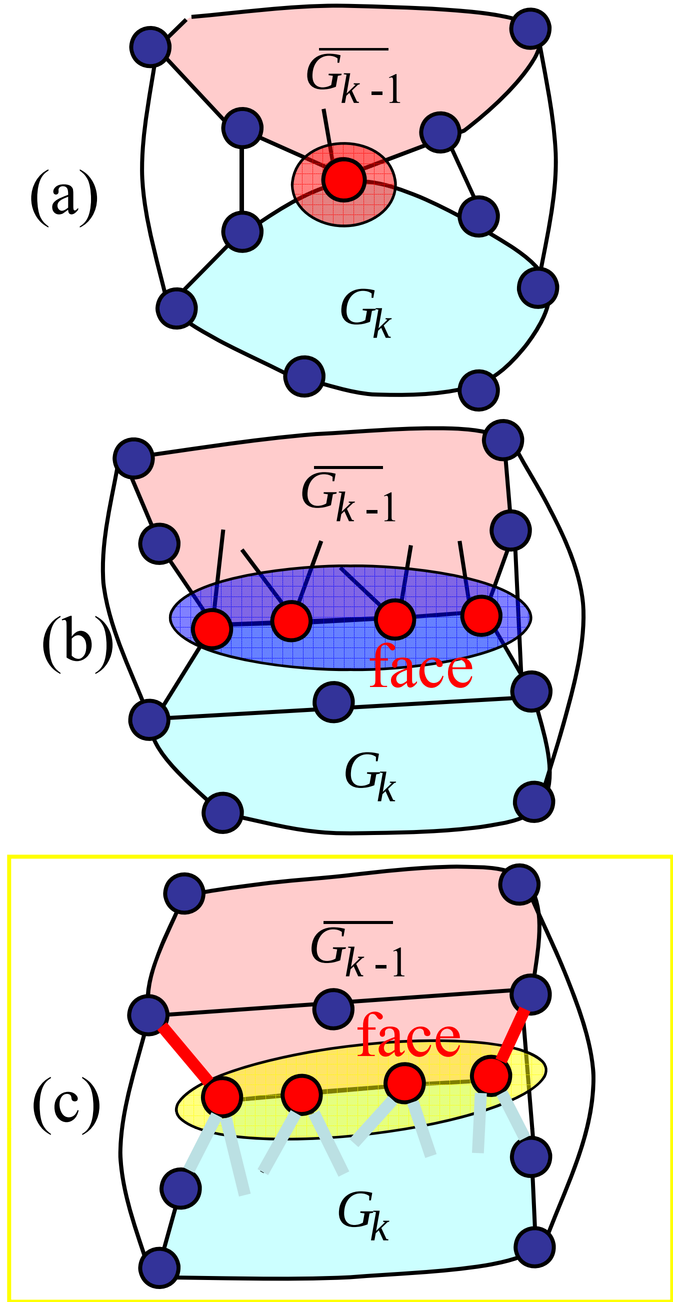




4-canonical decomposition[NRN97]

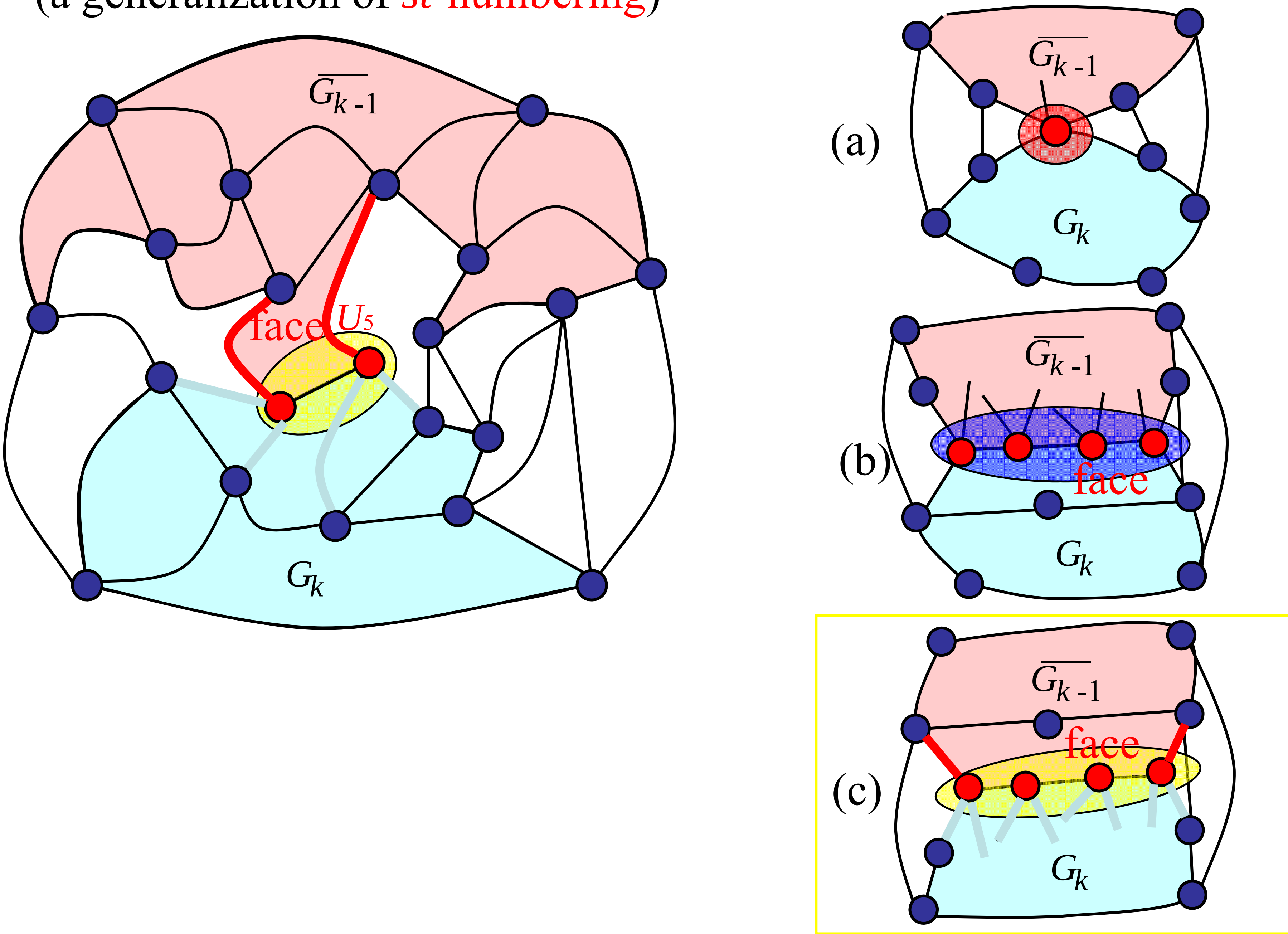
(a generalization of *st*-numbering)





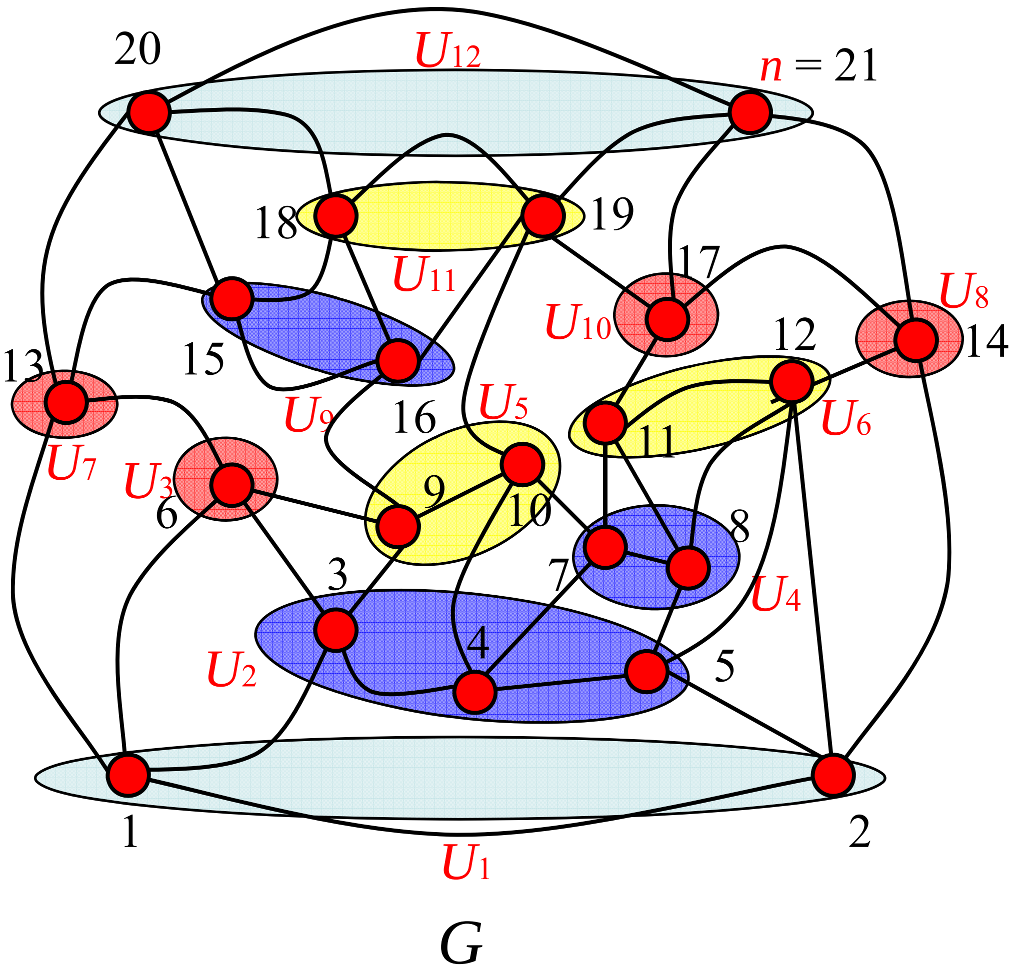
4-canonical decomposition[NRN97]

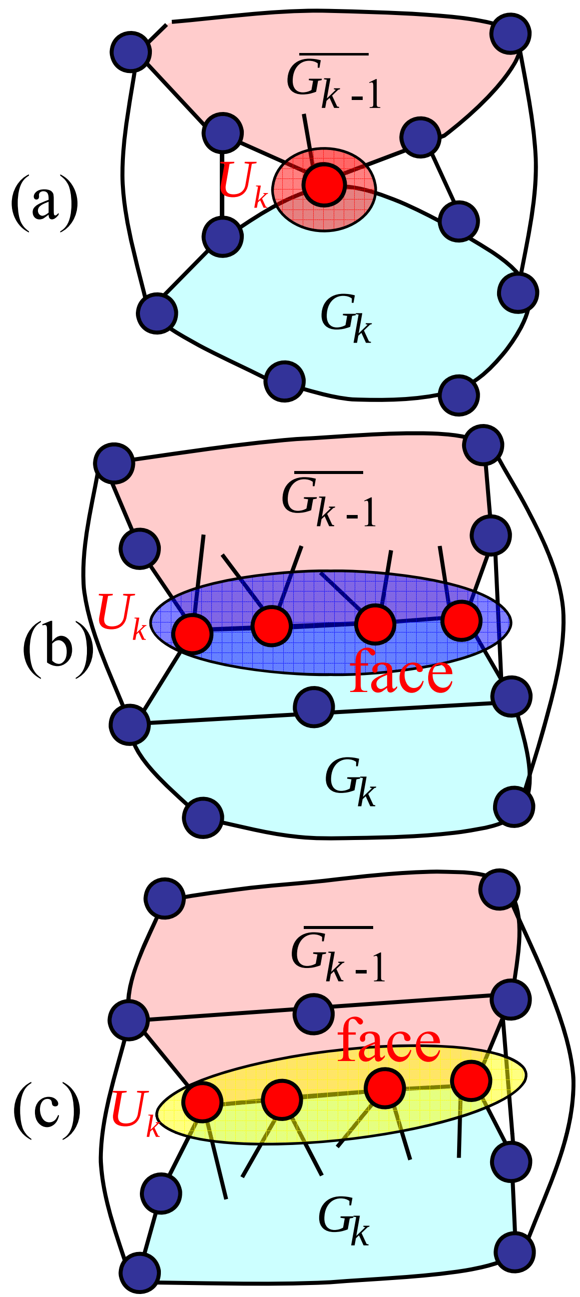
(a generalization of *st*-numbering)



4-canonical decomposition[NRN97] $O(n)$

(a generalization of *st*-numbering)





Conclusions

- *st*-numbering
- Canonical ordering
- 4-canonical ordering
- Canonical decomposition
- 4-canonical decomposition

Conclusions

Recent Development

Chiang *et al.*, 2001 Orderly spanning trees

Miura *et al.* 2004

Canonical decomposition, realizer and orderly Spanning tree are equivalent notions.

Thank You